Discrete-Time signals and systems

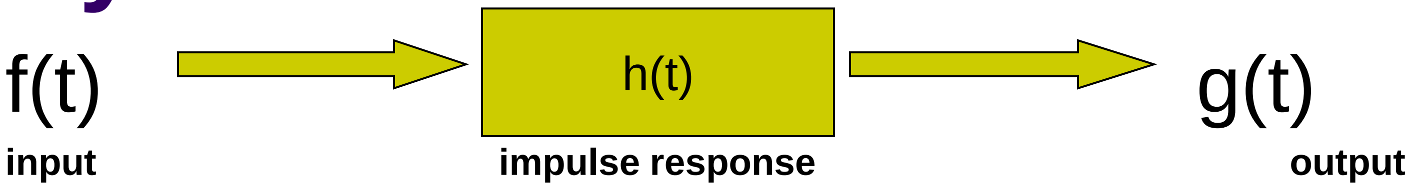
Introduction

- **Signal:** A signal can be defined as a function that conveys information, generally about the state or behavior of a physical system.

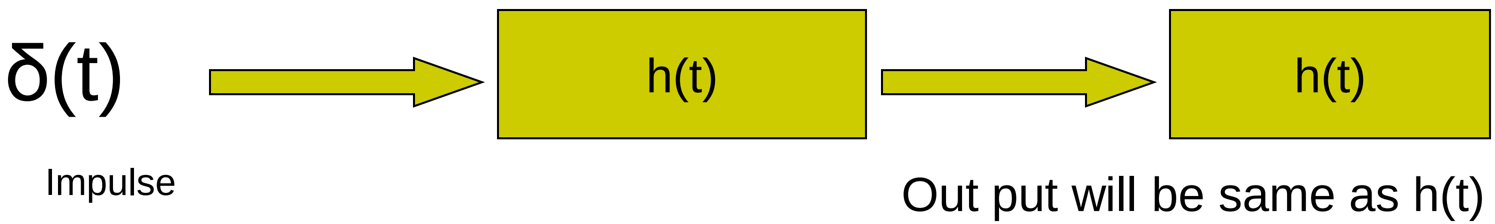
- **Continuous-time signal:** Continuous-time signals are defined along a continuum of times and thus are represented by a continuous independent variable. Continuous-time signals are often referred to as analog signals.

- **Discrete time signal:** Discrete-time signals are defined at discrete times and thus the independent variable has discrete values; i.e., discrete-time signals are represented as sequences of number.

Analog: Linear time invariant system



- **How to retrieve h(t):**



How to define δ(t) ?

- δ(t)= $\begin{cases} \alpha & t = 0 \\ 0 & Others \end{cases}$

And $$\int_{-\varepsilon}^{+\varepsilon} \partial(t)\,dt = 1$$

How to define System Output g(t)?

- g(t)= $$\int_{-\infty}^{+\infty} h(\tau).f(t-\tau)d\tau$$

- Or g(t)= $$\int_{-\infty}^{+\infty} f(\tau).h(t-\tau)d\tau$$

- And there is another a form like

  g(t)=h(t)*f(t)

Here * is the commutative property of convolution

Frequency domain:

- In frequency domain system output

$$G(\omega)= \int_{-\infty}^{+\infty} g(t)e^{-j\omega t}dt$$

G(ω)=H(ω) x F(ω)

Definition: Important functions in Discrete Time Signal

- Unit impulse function:

  δ(n) = $\begin{cases} 1 & for, n = 0 \\ 0 & for, n \neq 0 \end{cases}$

Unit step function:

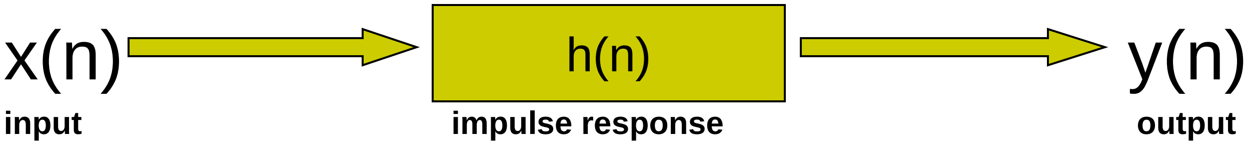
U(n)= $\begin{cases} 1 & for, n \geq 0 \\ 0 & for, n < 0 \end{cases}$

Here “n” is an integer

Relationship between δ(n) and U(n)

- U(n)= $$\sum_{k=-\infty}^{n} \partial(k)$$
- U(n)= $$\sum_{k=0}^{\infty} \partial(n-k)$$
- δ(n) = U(n)-U(n-1)

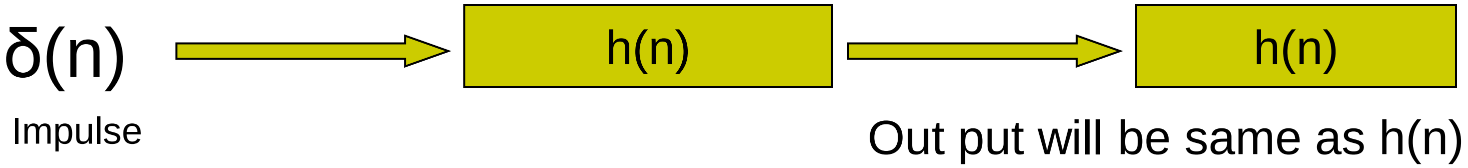
Discrete time invariant system



Output y(n) = $$\sum_{i=-\infty}^{i=+\infty} h(i)x(n-i)$$

Or y(n) = $$\sum_{i=-\infty}^{i=+\infty} x(i)h(n-i)$$

How to retrieve h(n)?

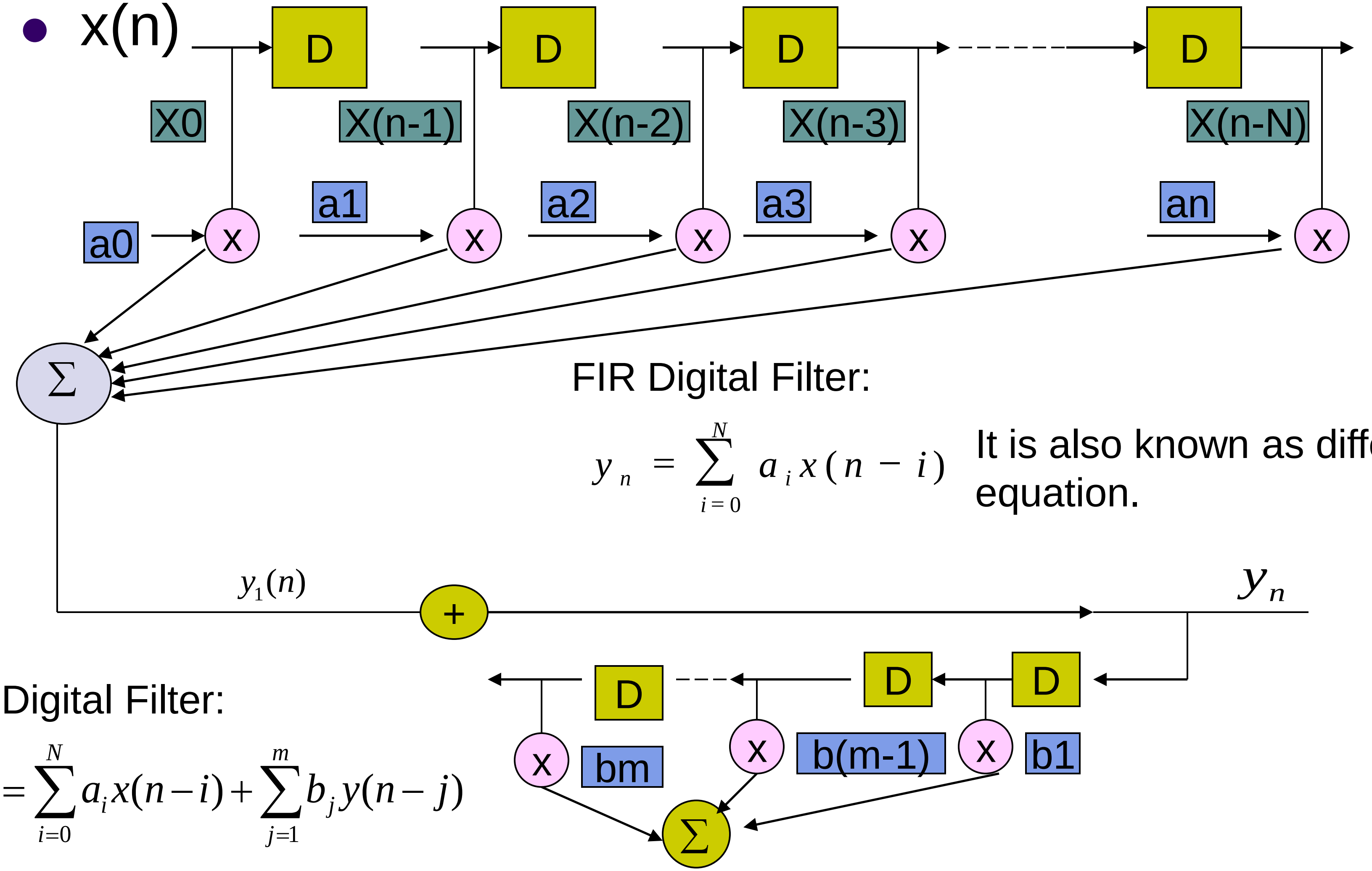


Definitions:

- **FIR (Finite Impulse Response):** A finite impulse response (FIR) filter is a type of a digital filter. The impulse response, the filter's response to a Kronecker delta input, is 'finite' because it settles to zero in a finite number of sample intervals.
- **IIR(Infinite Impulse Response):** They have an impulse response function which is non-zero over an infinite length of time.
- **Casual System:** If $h(n)=0$ for $n<0$; then this system is called casual system.

Follow graph(FIR digital filter,IIR digital filter):

- x(n)



FIR Digital Filter:

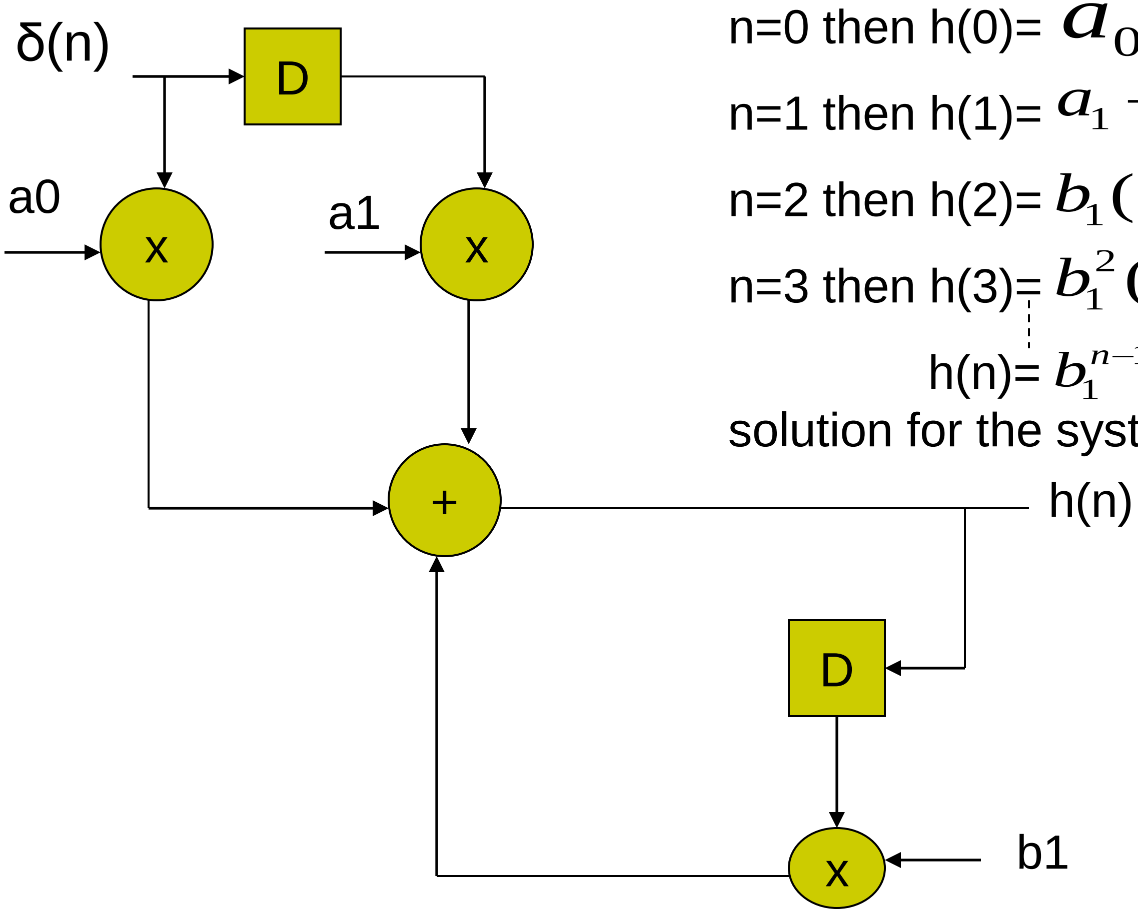
$$y_n = \sum_{i=0}^{N} a_i x(n-i)$$

It is also known as difference equation.

IIR Digital Filter:

$$y_n = \sum_{i=0}^{N} a_i x(n-i) + \sum_{j=1}^{m} b_j y(n-j)$$

The System Output h(n)



h(n)= $a_0\partial(n)+a_1\partial(n-1)+b_1h(n-1)$

n=0 then h(0)= a_0

n=1 then h(1)= $a_1+b_1a_0$

n=2 then h(2)= $b_1(a_1+b_1a_0)$

n=3 then h(3)= $b_1^2(a_1+b_1a_0)$

h(n)= $b_1^{n-1}(a_1+b_1a_0)$ it is general solution for the system output h(n)

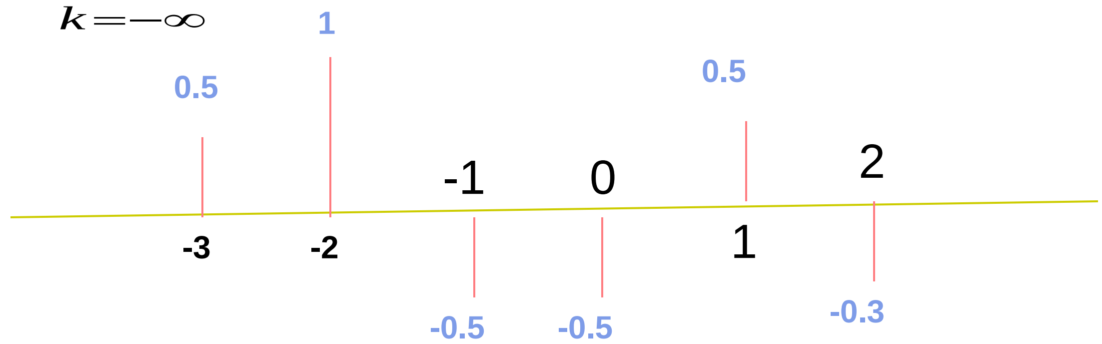
The energy of a sequence:

$$E=\sum_{n=-\infty}^{+\infty}|x(n)|^2$$

Any discrete sequence can be shown by δ(n) :

General equation: $x(n)=\sum_{k=-\infty}^{\infty}x(k)\partial(n-k)$

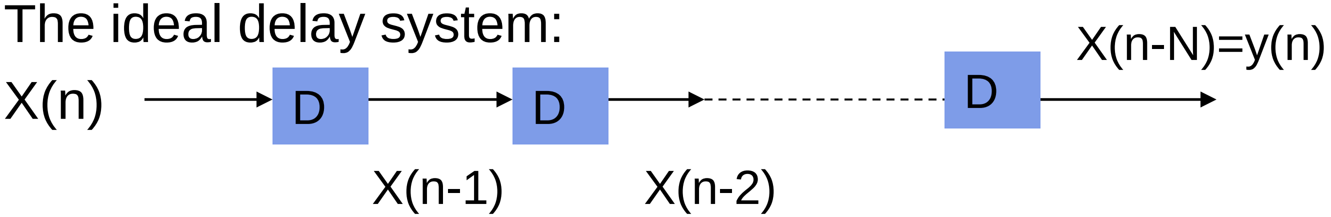
Example:



X(n)=0.5δ(n+3)+ δ(n+2)-0.5δ(n+1)-0.5 δ(n)+0.5 δ(n-1)-0.3δ(n-2)

Definition: Different System

The ideal delay system:



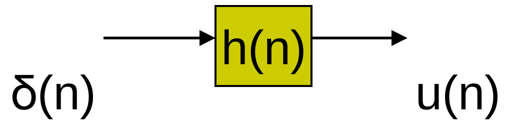
The ideal delay system output would be y(n)=x(n-N)

Moving Average: The moving average system output

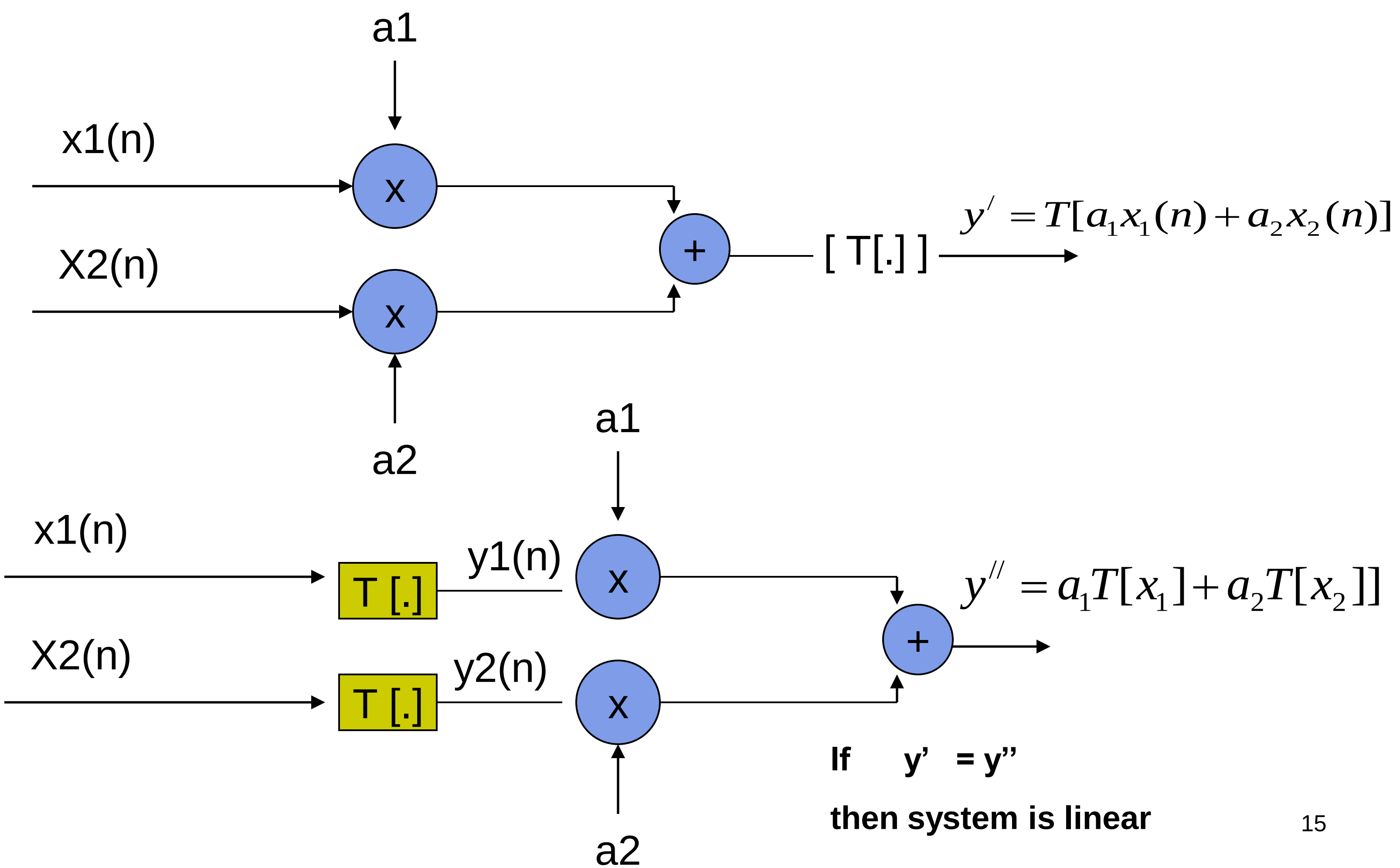
$$y(n)=\frac{1}{m1+m2+1}\sum_{k=-m1}^{m2} x(n-k)$$

Accumulator (Acc) : $y(n)=\sum_{k=-\infty}^{n} x(n)$

So the impulse response of the accumulator is same as the u(n)



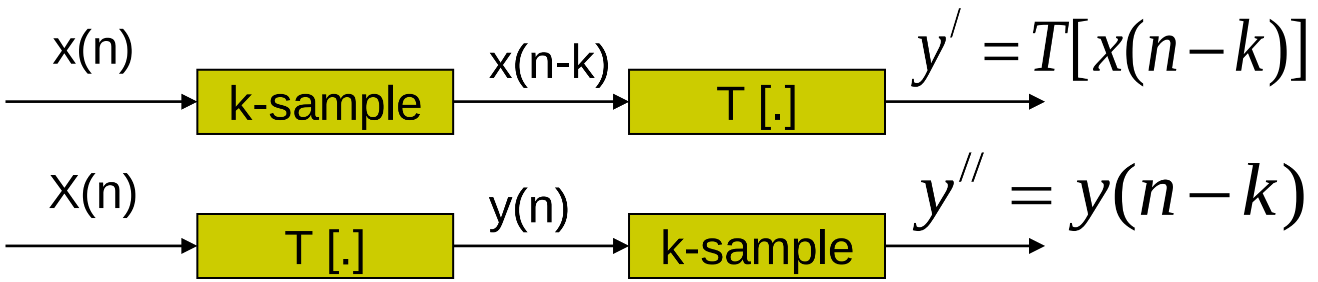
Linear Shift Invariant System



If y' = y''

then system is linear

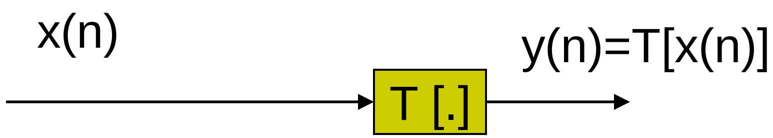
Linear Shift Invariant System (Cont...)



If $y^{//} = y(n-k)$ = T[x(n-k)] then the system is shift invariant

Discrete Convolution

- In LSI system



x(n)= $\sum_{k=-\infty}^{+\infty} x_{(k)} \partial_{(n-k)}$

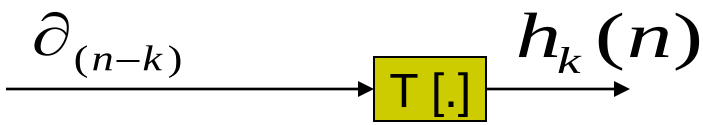
So, y(n)=T[$\sum_{k=-\infty}^{+\infty} x_{(k)} \partial_{(n-k)}$]

If the system is linear then

y(n) = $\sum_{k=-\infty}^{+\infty} x_{(k)} T[\partial_{(n-k)}]$

Discrete Convolution (Cont...)

- If the system is LSI



So we can write

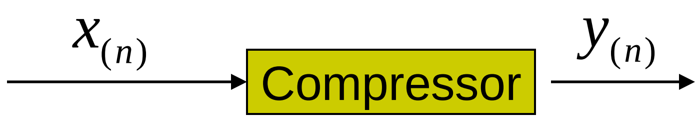
$$y(n)= \sum_{k=-\infty}^{+\infty} x_{(k)} h_{k(n)}$$

Again if the system is SI: $y_{(n-k)} = T[x_{(n-k)}]$ and $h_{k(n)} = h_{(n-k)}$

if LSI: $y(n)=\sum_{k=-\infty}^{+\infty} x_{(k)} h_{k(n-k)}$,it is known as discrete convolution

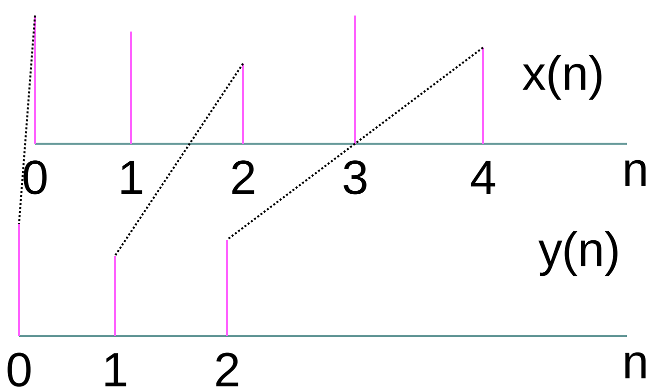
We can also write as $y(n)=\sum_{k=-\infty}^{+\infty} h_{(k)} x_{k(n-k)}$

Compressor: Down Sampling :Decimator



- The compressor output y(n)=x(M.n) where M is a integer greater than 1

If M=2;



Discarding M-1 in between samples

A compressor is not SI:

x1(n)=x(n-n0)

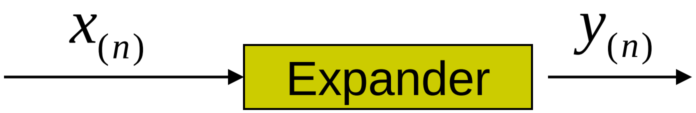
y1(n)=x1(mN)=x(Mn-n0)

y(n-n0)=x[M(n-n0)]=x(Mn-Mn0)

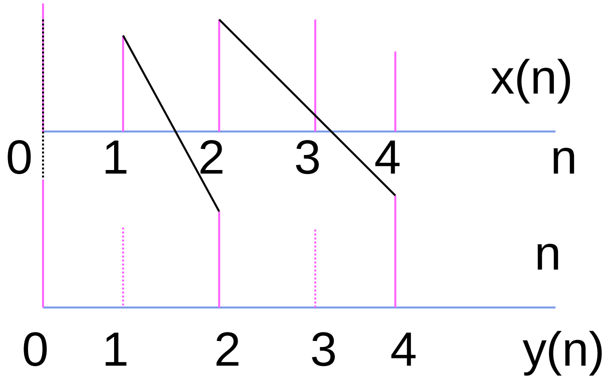
Here, x(Mn-n0)!=x(Mn-Mn0)

So, a compressor not SI

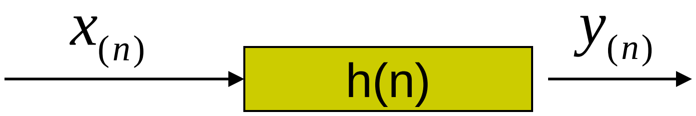
Expander : Up Sampling :Interpolation



- The expander output y(n)=x(n/L); where L>1
- If L=2;



The system output y(n) in different cases



- h(n)= $\begin{cases} a^n \; for, n \geq 0 \\ 0 \; for, n < 0 \end{cases}$

x(n)=U(n) - U(n-N)

y(n)=?

We know that system output for LSI: y(n)= $\sum_{k=-\infty}^{+\infty} x_{(k)} h_{(n-k)}$

1. For n<0 then y(n)=0
2. For 0<=n<N then y(n)= $\sum_{k=0}^{n} a^{n-k} = a^n \sum_{k=0}^{n} a^{-k} = a^n \frac{1-a^{-(n+1)}}{1-a^{-1}}$
3. For n>=N-1 then y(n)= $\sum_{k=0}^{N-1} a^{n-k} = a^n \frac{1-a^{-N}}{1-a^{-1}}$

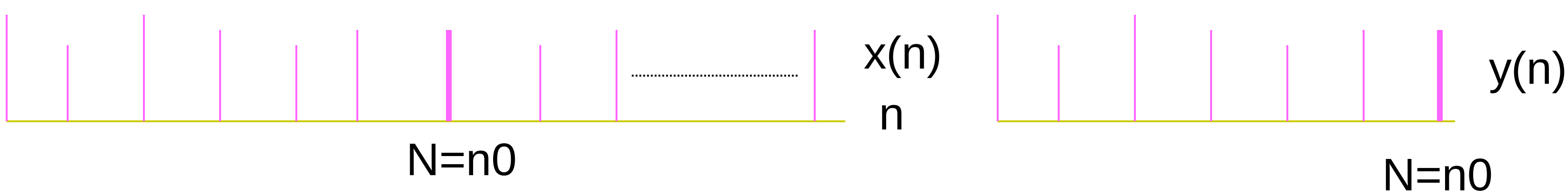
Definition:

- **Stability of a system**: A stable system produces finite output when the input is finite $|y(n)| = |\sum_{k=-\infty}^{+\infty} h_k x_{(n-k)}| < \infty$

For LSI system: and |x(n)|<M<∞

So, |y(n)|< $M \sum_{k=-\infty}^{+\infty} |h_{(k)}| < \infty \Rightarrow \sum_{k=-\infty}^{+\infty} |h_{(k)}| < \infty$

Causality: A system is casual when for N=n0; the output of the system depending on input x(n) only for n<=n0



If the system is LSI and h(n)=0 for n<0 then it is causal

Example: s= $\sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=0}^{\infty} |a^n| < \infty$

1.|a|<1;S=1/1-|a|;Stable

2.a=1 and a>1; s=∞ then not stable

Impulse response for different delay

Ideal Delay:

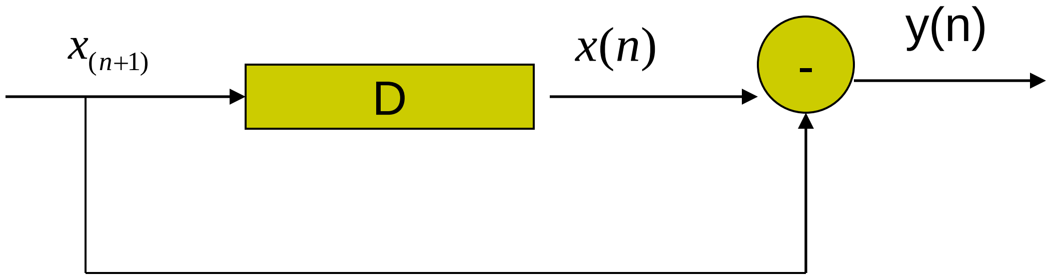
Impulse response for ideal delay h(n)=δ(n-m)

Moving average:

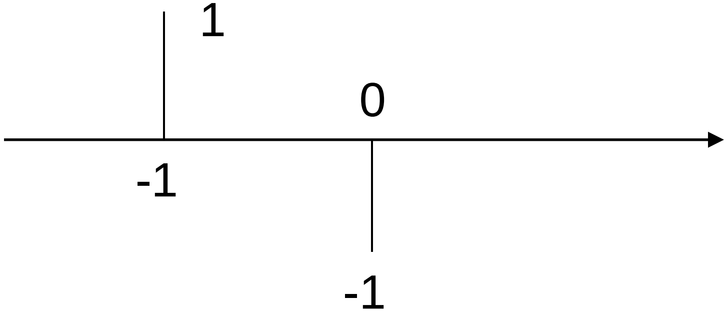
h(n)= $\frac{1}{m1+m2+1}\sum_{k=-m1}^{m2}\partial(n-k)=\begin{cases}\frac{1}{m1+m2-1} for, n-m1\leq n\leq m2+n \\ 0 \ \ otherwise\end{cases}$

Accumulator: h(n)=u(n)= $\sum_{k=-\infty}^{n}\partial(k)=\begin{cases}1 \ \ n\geq 0 \\ 0 \ \ n<0\end{cases}$

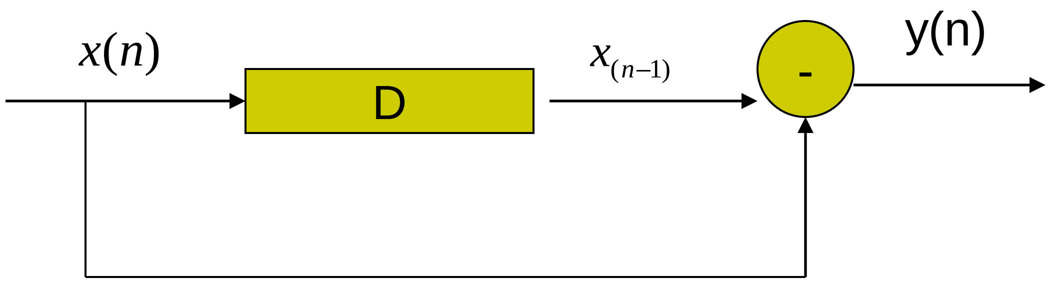
Forward difference:



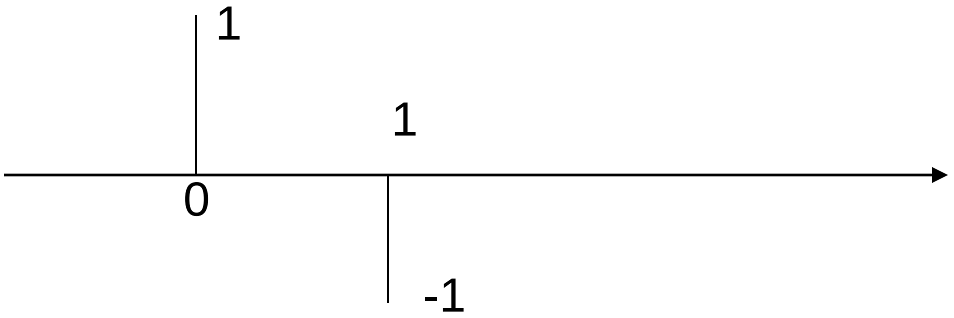
- Difference eqation = x(n+1)-x(n)
- Impulse response = δ(n+1)-δ(n)



Backward Difference



- Difference equation= y(n)=x(n)-x(n-1)
- Impulse response h(n)=δ(n+1)-δ(n)



Stability:

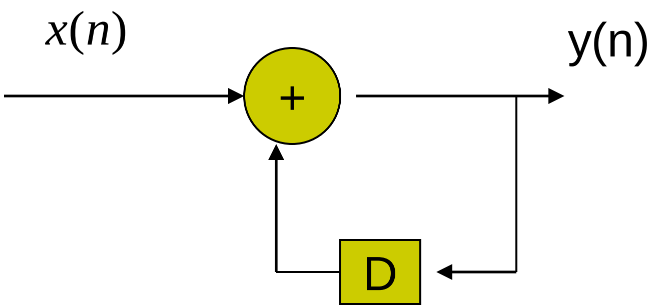
- For a LSI $s = \sum_{n=-\infty}^{+\infty} |h(n)| < \infty$

- For Ideal Delay, Moving Average, Backward Difference and Forward Difference; if s<∞ then it is stable

But for Accumulator

s=$\sum_{n=0}^{\infty} U(n)$ goes to ∞ then it is not stable

Accumulator

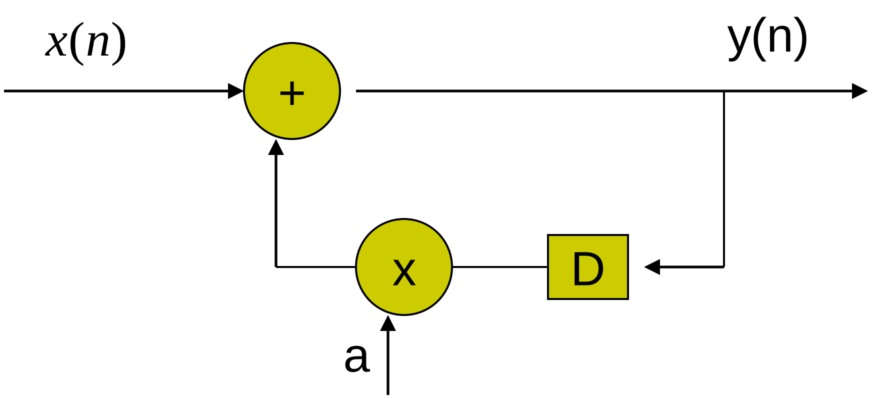
- 

Difference equation

y(n)=x(n)+y(n-1)

h(n)=U(n)

- **This IIR digital filter**



Difference equation

y(n)=x(n)+ay(n-1)

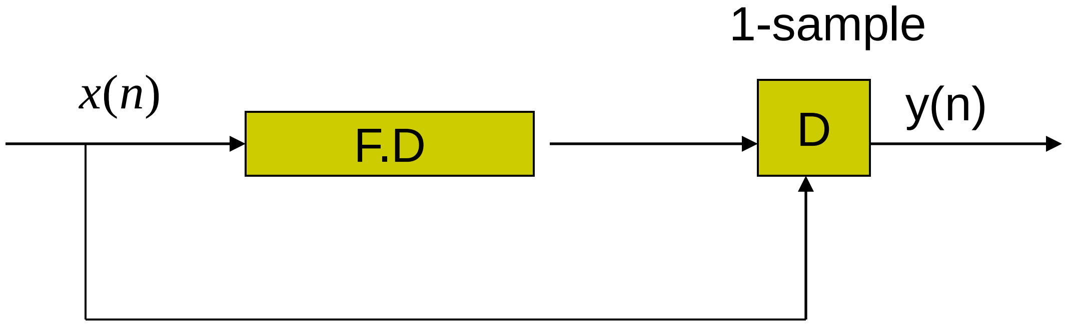
Impulse response $h(n)=a^n u(n)$

Stability checking $S=\sum|a^n|=\dfrac{1}{1-|a|}<\infty$

Condition check: if |a|<1

Result: Stable

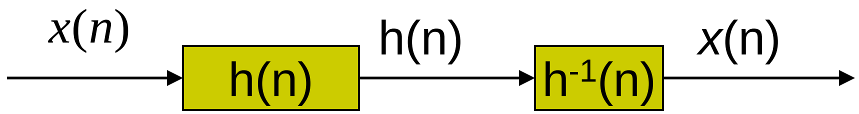
Other properties of LSI



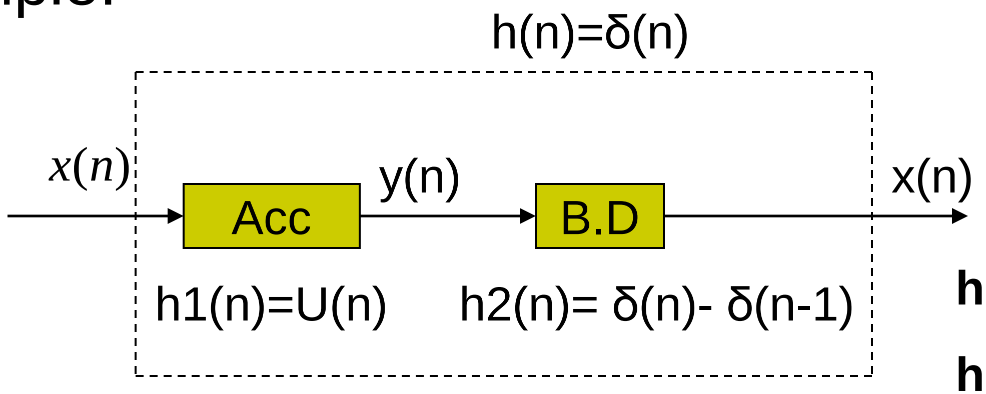
h(n)=[δ(n+1)-δ(n)]*δ(n-1)

h(n)=δ(n)- δ(n-1) ; In case of forward delay, if we add a 1-sample delay then it will convert to B.D

Inverse of a system

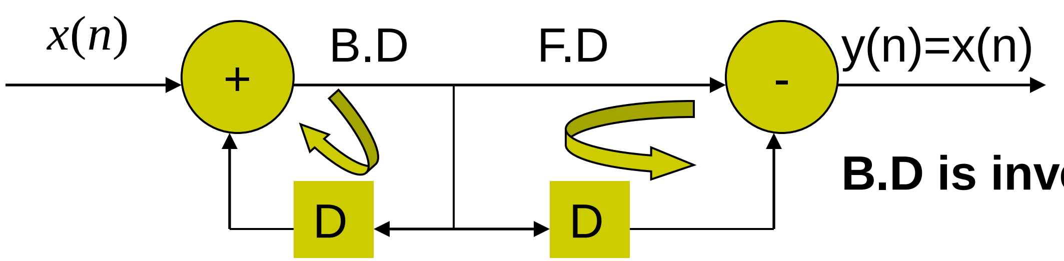
- 

Example:



h(n)=h1(n)*h2(n)

h(n)=U(n)*[δ(n)- δ(n-1)]=U(n)-U(n-1)= δ(n)



B.D is inverse system of ACC

Inverse of system: Engineering application

- 1. T.V.ghost canceling
- 2. Channel multi-path canceling
- 3. Equalization in communications channel

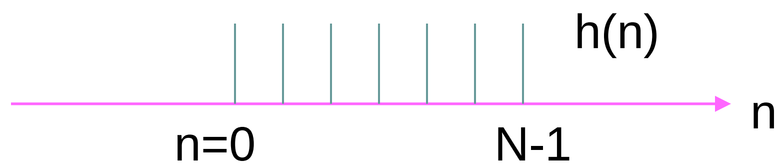
Frequency domain representation of discrete time signals and system

- **Frequency response of the system**:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} h(k) e^{-j\omega k}$$

Example:

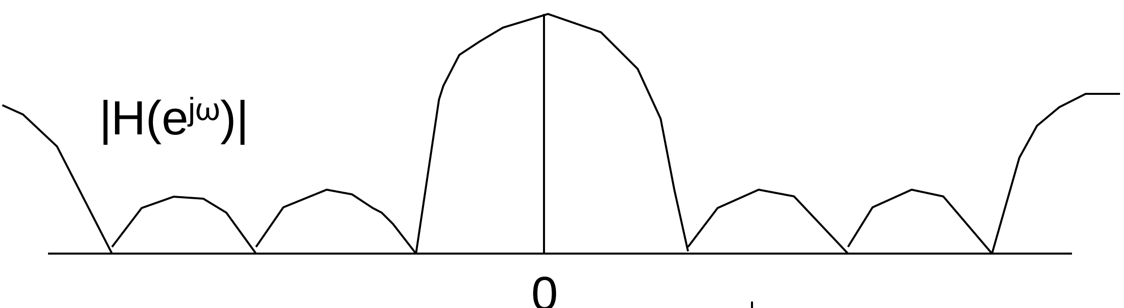
$$h(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & 0, elsewhere \end{cases}$$



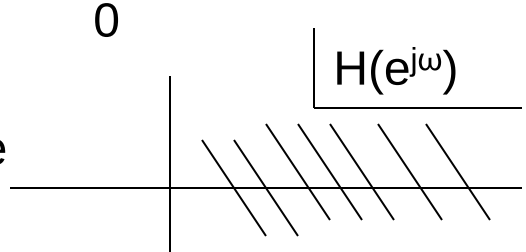
h(n)=U(n)-U(n-N)

$$H(e^{j\omega}) = \sum_{k=0}^{N-1} h(k) e^{-j\omega k} = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

$$= \frac{\sin(\frac{\omega N}{2})}{\sin(\frac{\omega}{2})} e^{-j\frac{N-1}{2}\omega}$$



For linear phase

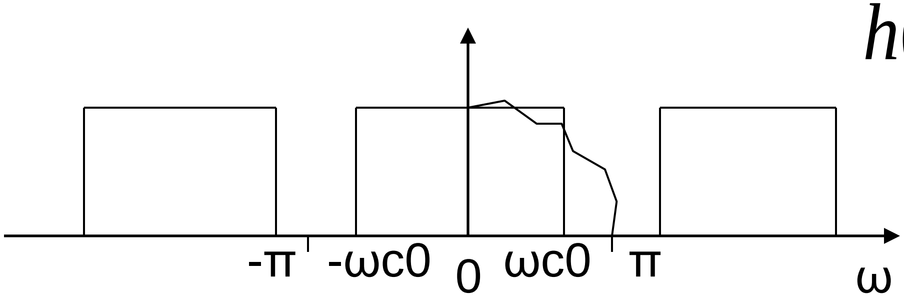


Inverse Fourier transform:

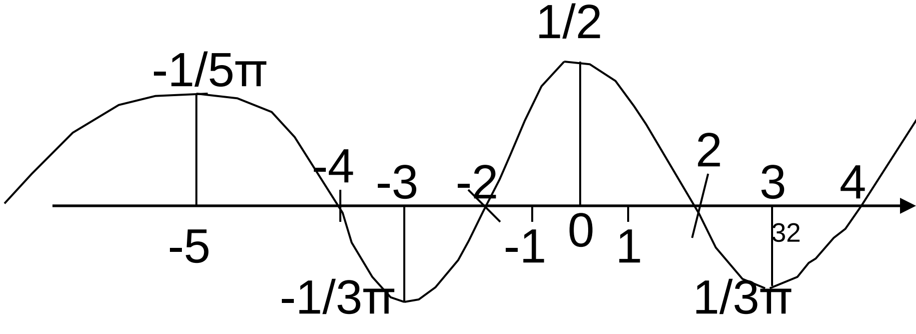
$$h(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H(e^{j\omega})e^{j\omega n}d\omega$$

Example: In the ideal low pass filter $H(e^{j\omega})=\begin{cases}1 & |\omega|\le\omega_{c0}\\ 0 & \omega_{c0}<|\omega|\le\pi\end{cases}$

$|h(e^{j\omega})|$



$$h(n)=\frac{1}{2\pi}\int_{-\omega_{c0}}^{\omega_{c0}}e^{j\omega N}d\omega=\frac{\sin(\omega_{co}n)}{\pi n}$$



Proof the inverse Fourier transform

$$x(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{(n)} e^{-j\omega n}$$

$$\because \quad x_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(e^{j\omega}) e^{j\omega n} d\omega$$

$$\hat{x} \quad such \quad that: \ \hat{x}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{m=-\infty}^{+\infty} x_{(m)} e^{-j\omega m} \right) e^{j\omega n} d\omega$$

interchanging the order of integral and summation

$$\hat{x}(n) = \sum_{m=-\infty}^{+\infty} x_{(m)} \cdot \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-m)} d\omega \right]$$

Proof the inverse Fourier transform cont..

We have,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-\omega)}d\omega = \frac{\sin[\pi(n-m)]}{\pi(n-m)} = \begin{cases} 1 & for, m=n \\ 0 & m\neq n \end{cases}$$

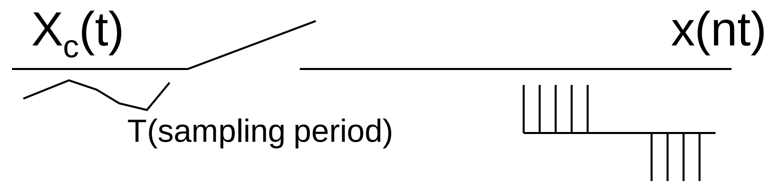
=δ(n)

$$\hat{x}(n) = \sum_{m=-\infty}^{+\infty} x(m)\delta(n-m)$$

$$\hat{x}(n) = x(n)$$

Sampling theorem:

Analog signal $x_c(t)$



FT: $$x_c(\Omega) = \int_{-\infty}^{+\infty} x_c(t) e^{-j\Omega t} dt \ldots\ldots\ldots\ldots\ldots(1)$$

IFT: $$x_c(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} x_c(\Omega) e^{j\Omega t} d\Omega \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Fourier transform for discrete time signal:

Digital freq: $\omega = \Omega . T$……………………………………….(3)

$$x(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(nT)e^{-jn\omega} = \sum_{n=-\infty}^{+\infty} x(nT)e^{-jn\Omega T} \text{............}(4)$$

From eq.(2) for t=n.T

$$x(nT) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x_c(\Omega)e^{j\omega nT}d\Omega \text{........................}(5)$$

On the other hand we had before:

$$x(nT) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} x(e^{j\omega})e^{j\omega n}d\omega \text{........................}(6)$$

Putting (5) into (4)

$$x(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} x_c(\Omega) e^{j\Omega nT} d\Omega] e^{-jn\omega} \ldots\ldots\ldots\ldots\ldots(7)$$

Interchanging ∑ with integral

$$x(e^{j\omega}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x_c(\Omega) \left[\sum_{n=-\infty}^{\infty} e^{jn(\Omega T - \omega)} \right] d\Omega \ldots\ldots\ldots\ldots\ldots(8)$$

But we have:

$$\sum_{n=-\infty}^{+\infty} e^{jn(\Omega T-\omega)} = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\partial(\Omega-\frac{\omega}{T}+\frac{2\pi k}{T})..............(9)$$

Then from (8) and (9) :

$$x(e^{j\omega}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x_c(\Omega).\left[\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\partial(\Omega-\frac{\omega}{T}+\frac{2\pi k}{T})\right]d\Omega$$

$$x(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_c(\Omega)\partial(\Omega-\frac{\omega}{T}+\frac{2\pi k}{T})d\Omega.........(10)$$

Then

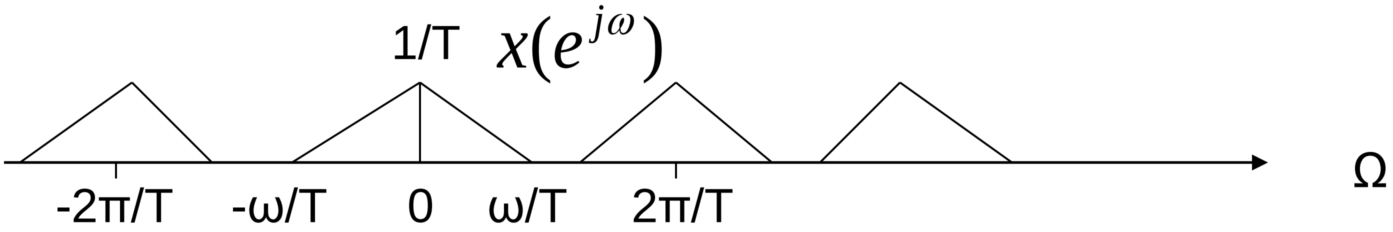
$$x(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} x_c\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)$$

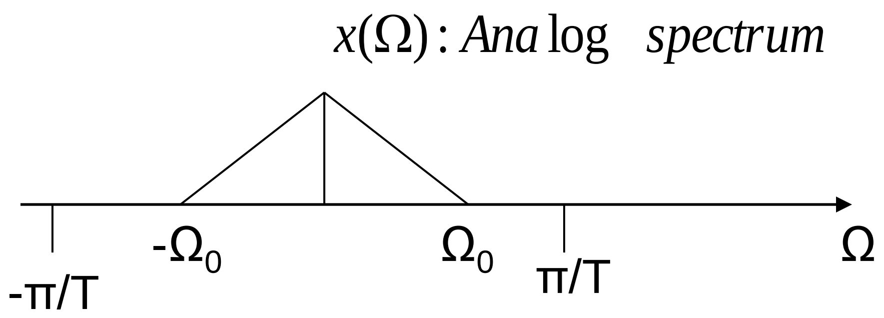
Since,

$$\Omega = \frac{\omega}{T}$$

$$x(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} x_c\left(\Omega - \frac{2\pi k}{T}\right)................(11)$$

Then $x(e^{j\omega})$ is a periodic function of Ω=ω/T with period of 2π/T





Nyquist rate for maximum frequency Ω_0 is sampling rate in order not to have aliasing effect

if $$\Omega_0 < \frac{\pi}{T} = \frac{\omega_s}{2} = \pi f_s$$

$$-\pi \le \omega \le \pi$$

In this case we can recover the analog signal from its sample as follows:

$$x_c(t) = \sum_{k=-\infty}^{+\infty} x_c(kt) \frac{\sin[\frac{\pi}{T}(t-kT)]}{[\frac{\pi}{T}(t-kT)]}$$

This formal is obtained as follows:

$$x_c(t) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} x_c(\Omega) e^{j\Omega t} d\Omega$$

$$-\frac{\pi}{T} \leq \Omega \leq \frac{\pi}{T}$$

$$x(e^{j\omega}) = x(e^{j\omega T}) = \frac{1}{T} x_c(\Omega)$$

$$x_c(t) = \frac{1}{2\pi} \int_{-\pi/T}^{+\pi/T} T.x(e^{j\omega}) e^{j\Omega t} d\Omega$$

$$x(e^{j\omega}) = x(e^{j\Omega T}) = \sum_{k=-\infty}^{+\infty} x(kT) e^{-j\Omega Tk}$$

$$x_c(t) = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \left[\sum_{k=-\infty}^{+\infty} x(kT) e^{-j\Omega Tk} \right] e^{j\Omega t} d\Omega$$

$$x_c(t) = \sum_{k=-\infty}^{+\infty} x(kT)\left[\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} e^{j\Omega(t-kT)} d\Omega\right]$$

$$x_c(t) = \sum_{k=-\infty}^{+\infty} x(kT)\frac{\sin\left[(\frac{\pi}{T})(t-kT)\right]}{(\frac{\pi}{T})(t-kT)}$$

To recover analog signal from its sample

Fourier transform properties

$$x(n) \xrightarrow{f} X(e^{j\omega})$$

1. Time shift: $x(n-M) \xrightarrow{f} e^{-j\omega M} X(e^{j\omega})$

2. Frequency shift $e^{j\omega_0 n} x(n) \xleftarrow{f} X(e^{j(\omega-\omega 0)})$

3. Time reversal: $x(-n) \xrightarrow{f} X(e^{-j\omega})$

if x(n) is a real sequence then:

$$x(-n) \xrightarrow{f} X.(e^{j\omega})$$

Fourier transform properties contd..

4. Differentiations in frequency domain:

$$n.x(n) \longrightarrow j\frac{dx(e^{j\omega})}{d\omega}$$

5. Convolution theorem:

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)h(n-k)$$

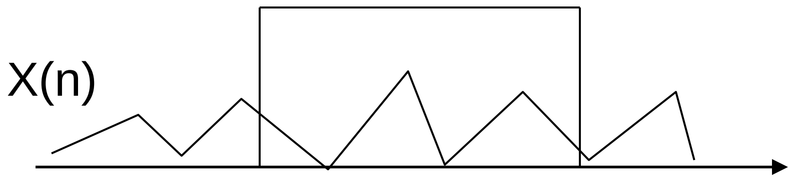
$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$

Fourier transform properties contd..

6. Parseval's Theorem (Energy)

$$E = \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x(e^{j\omega})|^2 d\omega$$

7. The modulation or windowing theorem:



y(n)=w(n) . x(n)

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta}).w(e^{j(\omega-\theta)})d\theta$$

Z-Transform

- X(z)= $\sum_{n=-\infty}^{\infty} x(n).z^{-n}$

z: complex variable in z-plane

Similar to Laplace transform

Convergency of Z-transfer should be check

Example 1. $x(n)=a^{n}U(n)$

X(z)= $\sum_{n=0}^{\infty} a^{n} z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^{n}$ From the geometric series if we have $|az^{-1}|<1 \Rightarrow |z|>|a|$

$x(z)=1/1-az^{-1}$

Example

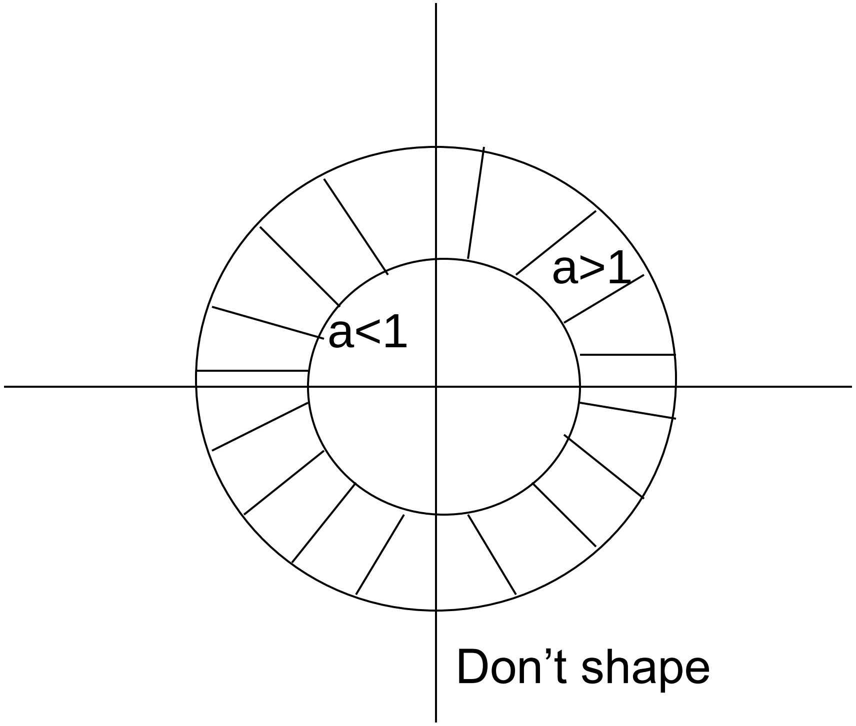
- $x(n)=a^{|n|}$

  $|a|<1$

$$X(z)=\sum_{n=1}^{\infty} a^{-n} z^{-n}+\sum_{n=0}^{\infty} a^{n} z^{-n}$$

$$=\sum_{n=1}^{\infty} a^{n} z^{n}+\sum_{n=0}^{\infty} a^{n} z^{-n}$$

1 2



1- convergent for $|az|<1$

2-convergent for $|az^{-1}|<1$

where $|a|<|z|<1/|a|$

Example contd…

So,

X(z)= $\dfrac{az}{1-az}+\dfrac{1}{1-az^{-1}}$

$$Z(z)=\frac{1-a^{z}}{(1-az)(1-az^{-1})}$$

Example

- Example 2:

  $x(n)=\delta(n)$

  $X(z)=1$

  $x(n)=\delta(n-m)$

  $X(z)=z^{-m}$

Example 3: $x(n)=a^n \sin(\omega_0 n)U(n)$

$$x(z)=\frac{a\sin(\omega_0)z^{-1}}{1-2a\cos\omega_0 z^{-1}+a^2 z^{-2}}$$

Example

- Example 4

$x(n)=na^nU(n)$

$$X(z)=\sum_{n=0}^{\infty} n a^n z^{-n}$$

With a little change in above summation

$$X(z)=z^{-1}\frac{d}{dz^{-1}}\left[\sum_{n=0}^{\infty} a^n z^{-n}\right]=z^{-1}\frac{d}{dz^{-1}}\left[\frac{1}{1-az^{-1}}\right]$$

$$X(z)=\frac{az^{-1}}{\left(1-az^{-1}\right)^2}$$

Properties of z-transform

- 1. Linearity: $a_1x_1(n)+a_2x_2(n) \xrightarrow{\text{Z-trans}} a_1X_1(z)+a_2X_2(z)$

- 2. Shift: $x(n \pm k) \longrightarrow Z^{\pm k} \; X(z)$

- 3 Convolution: $y(n)= \sum_{k=-\infty}^{+\infty} h(k)x(n-k)$

$$Y(Z)=h(Z).X(Z)$$

The relation between Z transform to Fourier transform

- $X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$

If we put: $z=e^{j\omega}$ then Z.T $\longrightarrow$ F.T

For more general case $Z=re^{j\omega}$

So, $$X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} \left[x(n) r^{-n} \right] e^{-j\omega n}$$

z-transform derived from Laplace transform

Consider a discrete-time signal x(t) below sampled every T sec

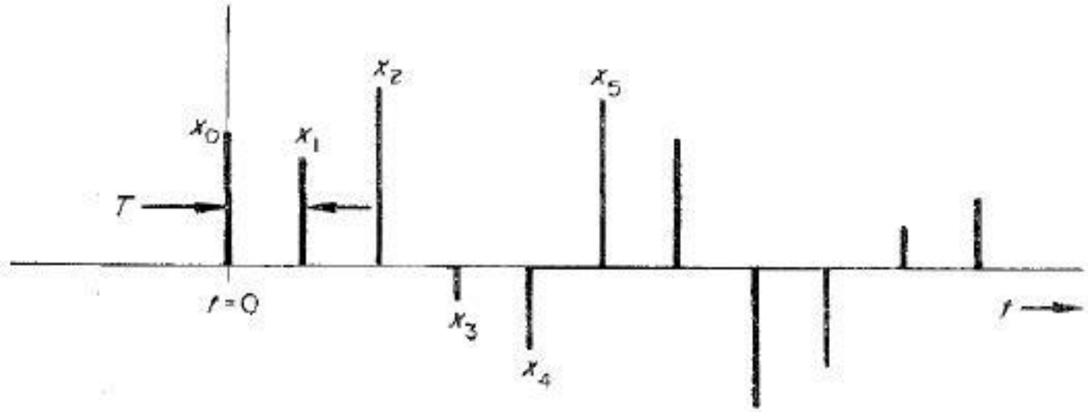
$x(t) = x_0\,\delta(t) + x_1\delta(t - T) + x_2\,\delta(t - 2T) + x_3\,\delta(t - 3T) + ...$

The Laplace transform of x(t) is therefore:

$X(s) = x_0 + x_1\, e^{-sT} + x_2\, e^{-s\,2T} + x_3\, e^{-s\,3T} + ...$

Now define $z = e^{sT} = e^{(\sigma + j\omega)T} = e^{\sigma T}\cos\omega T + je^{\sigma T}\sin\omega T$

$$X[z] = x_0 + x_1 z^{-1} + x_2 z^{-2} + x_3 z^{-3} + ...$$



Range of convergency (ROC)

- $x(n)=u(n)$

  in case $z=e^{j\omega}$ not convergent

  in case $z=re^{j\omega}$ for r>1 then convergent because

$$\sum_{n=-\infty}^{+\infty} | x(n) r^{-n} | < \infty$$

$$= \sum_{n=-\infty}^{+\infty} | u(n) r^{-n} | < \infty$$

$$= \sum_{n=0}^{\infty} | r^{-n} | < \infty$$

ROC Contd….

- Step function u(n) has z-transform for ROC: |z|>1

∵ If ROC includes the unit circle then $z=e^{j\omega}$ and the sequence has Fourier Transform

∵ There is possibility that two sequences are different but they may have a similar algebraic form of their z-transform, however their ROC's are different

Table of z-transform

Here:

- u[n]=1 for n>=0, u[n]=0 for n<0
- δ[n] = 1 for n=0, δ[n] = 0 otherwise

	Signal, $x[n]$	Z-transform, $X(z)$	ROC
1	$\delta[n]$	1	all z
2	$\delta[n-n_0]$	z^{-n_0}	$z \neq 0$
3	$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
4	$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
5	$nu[n]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\|z\| > 1$
6	$-nu[-n-1]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\|z\| < 1$
7	$n^2u[n]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\|z\| > 1$
8	$-n^2u[-n-1]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\|z\| < 1$
9	$n^3u[n]$	$\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$	$\|z\| > 1$
10	$-n^3u[-n-1]$	$\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$	$\|z\| < 1$

Table of z-transform

11	$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
12	$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
13	$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
14	$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
15	$n^2 a^n u[n]$	$\frac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}$	$\|z\| > \|a\|$
16	$-n^2 a^n u[-n-1]$	$\frac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}$	$\|z\| < \|a\|$
17	$\cos(\omega_0 n)u[n]$	$\frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\|z\| > 1$
18	$\sin(\omega_0 n)u[n]$	$\frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\|z\| > 1$
19	$a^n \cos(\omega_0 n)u[n]$	$\frac{1-az^{-1}\cos(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\|z\| > \|a\|$
20	$a^n \sin(\omega_0 n)u[n]$	$\frac{az^{-1}\sin(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\|z\| > \|a\|$

The properties of ROC

- ROC has a ring form or a disc form
- The Fourier transform of x(n) has Fourier transform if and only if that its z-transform's ROC includes unit circle
- ROC cannot contain any pole
- If the sequence x(n) has finite length then ROC contains all z-plane (excluding $z=0$ or $z=\infty$)
- If x(n) is right-sided, then ROC is located outside of the largest pole.
- If x(n) is left sided then ROC is located inside of the smallest pole.
- If the sequence x(n) is both-sided then the ROC has ring shape which is limited to inside and outside poles and there is no pole in ROC.
- ROC must be a connected area.

Calculation of inverse Z-transform

$$x(n) = \frac{1}{2\pi j} \oint_c x(z) z^{n-1} dz$$

C is a closed curve

Example: $$X(z) = \frac{1}{1 - 0.5z^{-1}}$$

|z|>0.5

$$X(z) = \frac{1}{1 - 0.5z^{-1}}$$ $=1+0.5z^{-1}+0.25z^{-2}+0.125z^{-3}$

$$x(n) = \begin{cases} 1, 0.5, 0.25, 0.125 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots n \geq 0 \\ 0 \qquad\qquad\qquad\qquad\qquad n<0 \end{cases}$$

$\Rightarrow x(n)=(0.5)^n u(n)$

Inverse z-transform

- If the Z transform can be expanded out as a series in powers of z, then the coefficients of each term of the series constitutes the inverse. In the following expression, the inverse would be the coefficients in blue

$$X_0(z) = x_0(0) + x_0(1)z^{-1} + x_0(2)z^{-2} + x_0(3)z^{-3} + \cdots$$

- Consider a Z transform which can be expanded as in the expression below

$$X(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{8}z^{-3} + \cdots$$

- Then the inverse is the coefficients in blue and can be written as follows.

$$x(k) = 2^{-k}$$

Laplace, Fourier and z-Tranforms

	Definition	Purpose	Suitable for ..
Laplace transform	$X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}dt$	Converts **integro-differential** equations to **algebraic** equations	Continuous-time system & signal analysis; stable or unstable
Fourier transform	$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$	Converts **finite time** signal to frequency domain	Continuous-time; stable system, convergent signals only; best for steady-state
Discrete Fourier transform	$X[n\omega_0]=\sum_{n=0}^{N_0-1} x[n]e^{jn\omega_0 T}$ N_0 samples, T = sample period $\omega_0 = 2\pi / T$	Converts finite **discrete-time** signal to **discrete frequency** domain	Discrete time, otherwise same as FT
z transform	$X[z]=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$	Converts **difference** equations into **algebraic** equations	Discrete-time system & signal analysis; stable or unstable