C H A P T E R

6

The Chaotic Motion of Dynamical Systems

We study simple deterministic nonlinear models which exhibit complex behavior.

6.1 INTRODUCTION

Most natural phenomena are intrinsically nonlinear. Weather patterns and the turbulent motion of fluids are everyday examples. Although we have explored some of the properties of nonlinear physical systems in Chapter 5, it is easier to introduce some of the important concepts in the context of ecology. Our goal will be to analyze the one-dimensional difference equation

$$x_{n+1} = 4rx_n(1 - x_n), \tag{6.1}$$

where x_n is the ratio of the population in the nth generation to a reference population. We shall see that the dynamical properties of (6.1) are surprisingly intricate and have important implications for the development of a more general description of nonlinear phenomena. The significance of the behavior of (6.1) is indicated by the following quote from the ecologist Robert May:

> " ... *Its study does not involve as much conceptual sophistication as does elementary calculus. Such study would greatly enrich the student's intuition about nonlinear systems. Not only in research but also in the everyday world of politics and economics we would all be better off if more people realized that simple nonlinear systems do not necessarily possess simple dynamical properties.*"

The study of chaos is currently very popular, but the phenomena is not new and has been of interest to astronomers and mathematicians for about one hundred years. Much of the current interest is due to the use of the computer as a tool for making empirical observations. We will use the computer in this spirit.

6.2 A SIMPLE ONE-DIMENSIONAL MAP

Many biological populations effectively consist of a single generation with no overlap between successive generations. We might imagine an island with an insect population that breeds in the summer and leaves eggs that hatch the following spring. Because the population growth occurs at discrete times, it is appropriate to model the population growth by *difference equations* rather than by differential equations. A simple model of density-independent growth that relates the population in generation $n + 1$ to the population in generation n is given by

$$P_{n+1} = aP_n, \tag{6.2}$$

where P_n is the population in generation n and a is a constant. In the following, we assume that the time interval between generations is unity, and refer to n as the time.

If $a > 1$, each generation will be a times larger than the previous one. In this case (6.2) leads to geometrical growth and an unbounded population. Although the unbounded nature of geometrical growth is clear, it is remarkable that most of us do not integrate our understanding of geometrical growth into our everyday lives. Can a

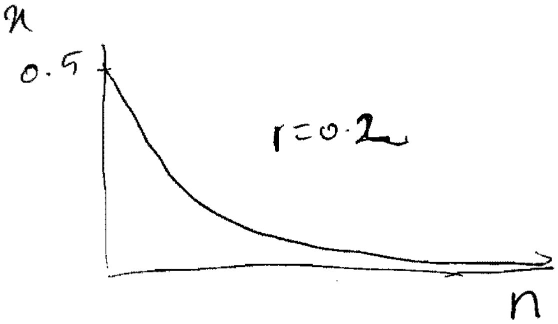
$$f(x) = 4rx(1-x)$$

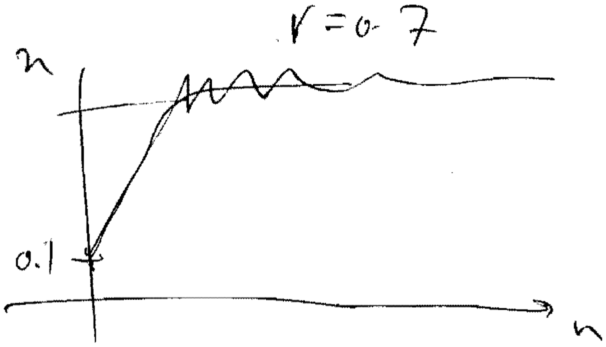
$$f^{(2)}(x) = 4r f(x)(1-f(x))$$

$$f^{(2)}(x) = 4r \times 4rx(1-x)\left(1-4rx(1-x)\right)$$

$$16r^2x(1-x)\left(1-4rx(1-x)\right)$$

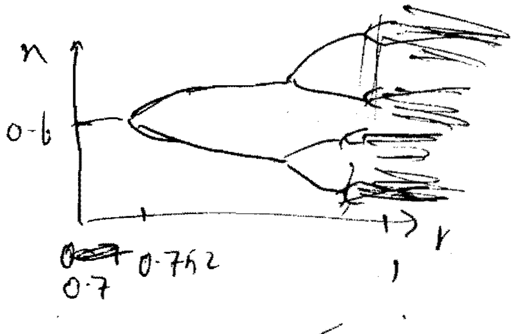
$$\overset{x}{[0\ \ 1]} \longrightarrow [0\ \ 1]$$



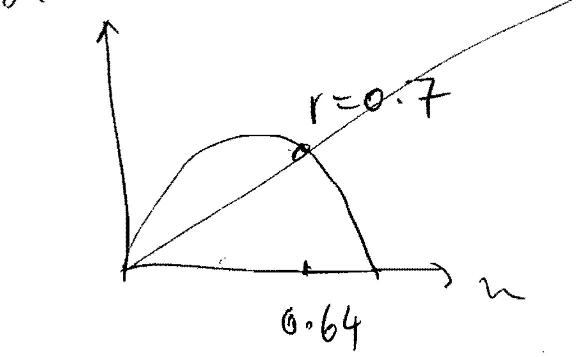


$r > 0.25$

$r > 0.75$



$$f(x) = \overset{4\times 0.7}{2.8}\,x(1-x)$$



$$\begin{array}{r} 1 - \\ 0.64 \\ \hline 0.36 \end{array}$$

bank pay 4% interest each year indefinitely? Can the world's human population grow at a constant rate forever?

It is natural to formulate a more realistic model in which the population is bounded by the finite carrying capacity of its environment. A simple model of density-dependent growth is

$$P_{n+1} = P_n(a - bP_n). \tag{6.3}$$

Equation (6.3) is nonlinear due to the presence of the quadratic term in P_n. The linear term represents the natural growth of the population; the quadratic term represents a reduction of this natural growth caused, for example, by overcrowding or by the spread of disease.

It is convenient to rescale the population by letting $P_n = (a/b)x_n$ and rewriting (6.3) as

$$x_{n+1} = ax_n(1 - x_n). \tag{6.4}$$

The replacement of P_n by x_n changes the system of units used to define the various parameters. To write (6.4) in the form (6.1), we define the parameter $r = a/4$ and obtain

$$x_{n+1} = f(x_n) = 4rx_n(1 - x_n). \tag{6.5}$$

The rescaled form (6.5) has the desirable feature that its dynamics are determined by a single control parameter r. Note that if $x_n > 1$, x_{n+1} will be negative. To avoid this unphysical feature, we impose the conditions that x is restricted to the interval $0 \le x \le 1$ and $0 < r \le 1$.

Because the function $f(x)$ defined in (6.5) transforms any point on the one-dimensional interval [0, 1] into another point in the same interval, the function f is called a one-dimensional *map*. The form of $f(x)$ in (6.5) is known as the *logistic* map. The logistic map is a simple example of a *dynamical system*, that is, a deterministic, mathematical prescription for finding the future state of a system.

The sequence of values $x_0, x_1, x_2, \cdots$ is called the *trajectory* or the *orbit*. To check your understanding, suppose that the initial condition or *seed* is $x_0 = 0.5$ and $r = 0.2$. Use a calculator to show that the trajectory is $x_1 = 0.2, x_2 = 0.128, x_3 = 0.089293, \ldots$ In Fig. 6.1 the first thirty iterations of (6.5) are shown for two values of r.

`Program iterate_map` computes the trajectory for the logistic map (6.5). The trajectory is listed in window 1 and plotted in window 2.

```
PROGRAM iterate_map                     ! iterate logistic map
CALL set_up_windows(#1,#2)
DO
   CALL initial(x,r,#1,#2,flag$)
   CALL map(x,r,#1,#2,flag$)
LOOP until flag$ = "stop"
END
```

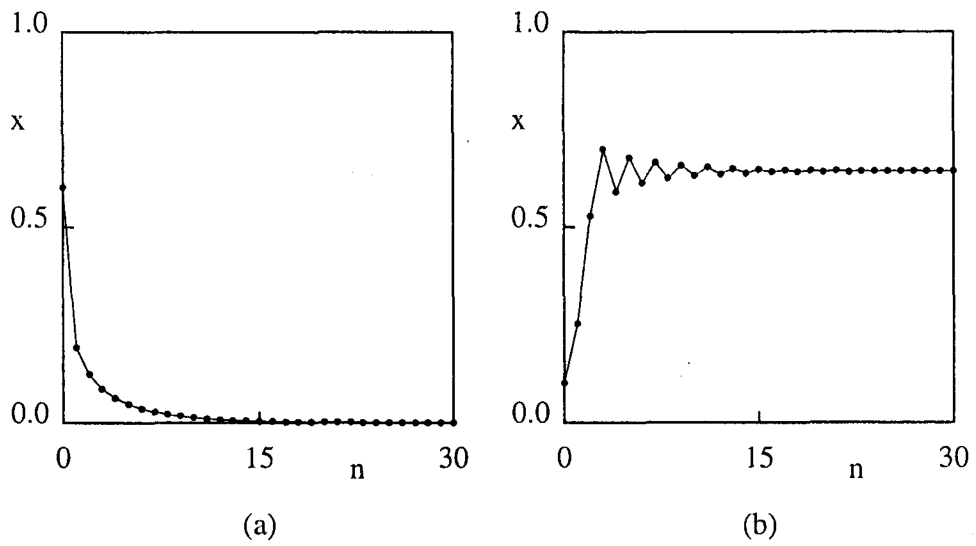


Fig. 6.1 (a) Time series for $r = 0.2$ and $x_0 = 0.6$. Note that the stable fixed point is $x = 0$. (b) Time series for $r = 0.7$ and $x_0 = 0.1$. Note the initial transient behavior. The lines between the points are a guide to the eye.

```
SUB initial(x0,r,#1,#2,flag$)
    WINDOW #2
    INPUT prompt "growth parameter (0 < r <= 1) = ": r
    LET x0 = 0.3
    CLEAR
    BOX LINES 0,1000,0,1
    SET CURSOR 1,2
    PRINT "r ="; r
    LET flag$ = ""
END SUB

SUB set_up_windows(#1,#2)
    OPEN #1: screen 0,1,0,0.5      ! text
    OPEN #2: screen 0,1,0.5,1      ! graphics
    LET nmax = 1000
    LET margin = 0.01*nmax
    SET WINDOW -margin,nmax+margin,-0.01,1.01
END SUB

SUB map(x,r,#1,#2,flag$)
    LET iterations = 0
    DO
       LET x = 4*r*x*(1 - x)        ! iterate map
       LET iterations = iterations + 1       ! number of iterations
       WINDOW #1
       SET COLOR "black/white"
```

```
      PRINT USING "#.######": x;
      ! period doubling implies convenient to start new line
      ! every 2^n iterations, where n = 2 or 3.
      IF mod(iterations,8) = 0 then PRINT  ! new line
      WINDOW #2
      SET COLOR "red"
      PLOT iterations,x
      IF key input then CALL change(#1,#2,flag$)
   LOOP until flag$ = "stop" or flag$ = "change"
   WINDOW #1
   PRINT
   PRINT "number of iterations = "; iterations
END SUB

SUB change(#1,#2,flag$)
   GET KEY k
   IF (k = ord("c")) or (k = ord("C")) then
      LET flag$ = "change"
      SET COLOR "black/white"
   ELSE IF (k = ord("s")) or (k = ord("S")) then
      LET flag$ = "stop"
   END IF
END SUB
```

In Problems 6.1 and 6.3 we use `Program map` to explore the dynamical properties of the logistic map (6.5). The program uses the `GET key` statement so that the key 'c' can be pressed to change the value of r and the key 's' can be pressed to stop the program.

Problem 6.1 Exploration of period-doubling

a. Explore the dynamical behavior of (6.5) with $r = 0.24$ for different values of x_0. Show that $x = 0$ is a *stable fixed point*. That is, for sufficiently small r, the iterated values of x converge to $x = 0$ independently of the value of x_0. If x represents the population of insects, describe the qualitative behavior of the population.

b. Explore the dynamical behavior of (6.5) for $r = 0.26, 0.5, 0.74$, and 0.748. A fixed point is *unstable* if for almost all x_0 near the fixed point, the trajectories diverge from it. Verify that $x = 0$ is an unstable fixed point for $r > 0.25$. Show that for the suggested values of r, the iterated values of x do not change after an initial *transient*, that is, the long time dynamical behavior is *period* 1. In Appendix 6A we show that for $r < 3/4$ and for x_0 in the interval $0 < x_0 < 1$, the trajectories approach the *attractor* at $x = 1 - 1/4r$. The set of initial points that iterate to the attractor is called the *basin* of the attractor. For the logistic map, the interval $0 < x < 1$ is the basin of attraction of the attractor $x = 1 - 1/4r$.

c. Explore the dynamical properties of (6.5) for $r = 0.752$, 0.76, 0.8, and 0.862. For $r = 0.752$ and 0.862 approximately 1000 iterations are necessary to obtain convergent results. Show that if r is increased slightly beyond 0.75, x oscillates between two values after an initial transient behavior. That is, instead of a stable cycle of period 1 corresponding to one fixed point, the system has a stable cycle of period 2. The value of r at which the single fixed point x^* splits or *bifurcates* into two values $x_1{}^*$ and $x_2{}^*$ is $r = b_1 = 3/4$. The pair of x values, $x_1{}^*$ and $x_2{}^*$, form a *stable attractor* of period 2.

d. Describe an ecological scenario of an insect population that exhibits dynamical behavior similar to that observed in part (c).

e. What are the stable attractors of (6.5) for $r = 0.863$ and 0.88? What is the corresponding period?

f. What are the stable attractors and corresponding periods for $r = 0.89$, 0.891, and 0.8922?

Another way to determine the behavior of (6.5) is to plot the values of x as a function of r (see Fig. 6.2). The iterated values of x are plotted after the initial transient behavior is discarded. Such a plot is generated by `Program bifurcate`. For each value of r, the first `ntransient` values of x are computed but not plotted. Then the next `nplot` values of x are plotted, with the first half in red and the second half in blue. This process is repeated for a new value of r until the desired range of r values is reached. A typical value of `ntransient` should be in the range of 100–1000 iterations. The magnitude of `nplot` should be at least as large as the longest period that you wish to observe.

```
PROGRAM bifurcate
! plot values of x for different values of r
CALL initial(x,r,rmax,nvalues,dr,ntransient,nplot)
FOR ir = 0 to nvalues
    CALL output(x,r,ntransient,nplot)
    LET r = r + dr
NEXT ir
! maximum value of r done separately to avoid r > 1
CALL output(x,rmax,ntransient,nplot)
END

SUB initial(x0,r0,rmax,nvalues,dr,ntransient,nplot)
    INPUT prompt "initial value of control parameter r = ": r0
    ! important that r not be greater than 1
    INPUT prompt "maximum value of r = ": rmax
    ! suggest dr <= 0.01
    INPUT prompt "incremental change of r = ": dr
    INPUT prompt "number of iterations not plotted = ": ntransient
    INPUT prompt "number of iterations plotted = ": nplot
    LET nvalues = (rmax - r0)/dr  ! number of r values plotted
    LET nvalues = int(nvalues)
```

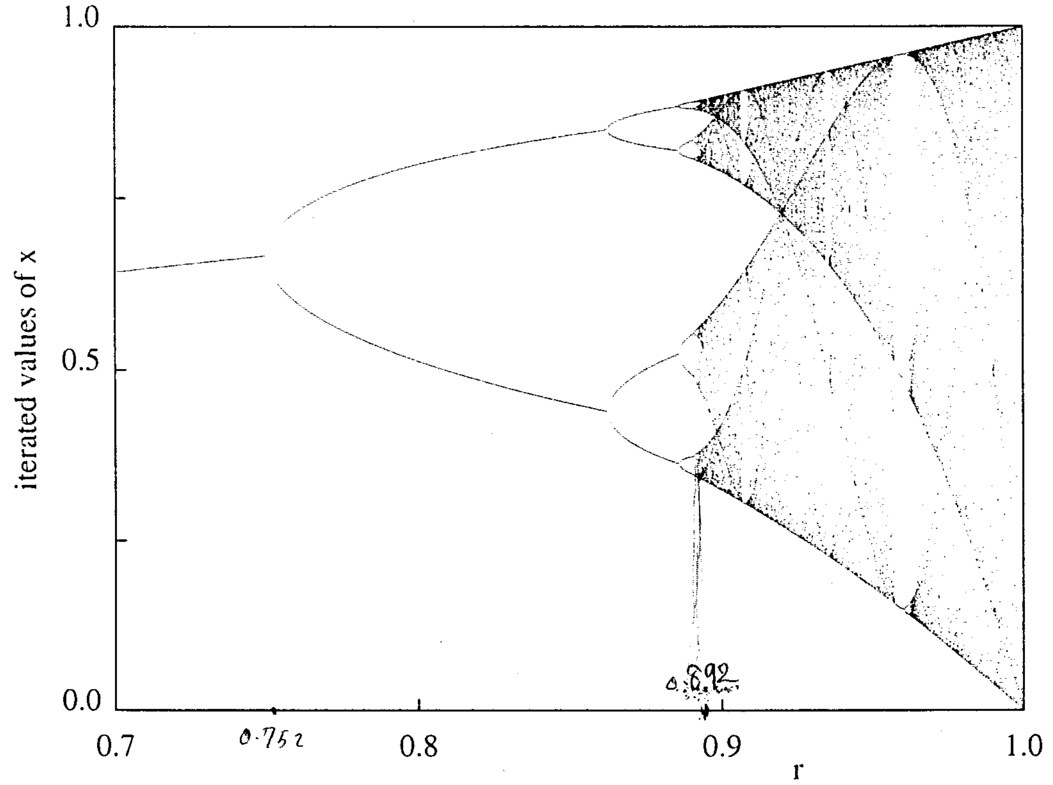


Fig. 6.2 Bifurcation diagram of the logistic map. For each value of r, the iterated values of x_n are plotted after the first 1000 iterations are discarded. Note the transition from periodic to chaotic behavior and the narrow windows of periodic behavior within the region of chaos.

The final state or bifurcation diagram in Fig. 6.2 indicates that the period-doubling behavior ends at $r \approx 0.892$. This value of r is known very precisely and is given by $r = r_\infty = 0.892486417967\ldots$ At $r = r_\infty$, the sequence of period-doublings accumulate to a trajectory of infinite period. In Problem 6.3 we explore the behavior of the trajectories for $r > r_\infty$.

Problem 6.3 The chaotic regime

a. For $r > r_\infty$, two initial conditions that are very close to one another can yield very different trajectories after a small number of iterations. As an example, choose $r = 0.91$ and consider $x_0 = 0.5$ and 0.5001. How many iterations are necessary for the iterated values of x to differ by more than ten percent? What happens for $r = 0.88$ for the same choice of seeds?

b. The accuracy of floating point numbers retained on a digital computer is finite. To test the effect of the finite accuracy of your computer, choose $r = 0.91$

```
    LET x0 = 0.5                      ! initial value
    CLEAR
    LET xmax = 1                      ! maximum value of x
    LET mx = 0.05*xmax                ! margin
    SET WINDOW r0-dr,rmax+dr,-mx,xmax + mx
    BOX LINES r0,rmax,0,1
END SUB

SUB output(x,r,ntransient,nplot)
    DECLARE DEF f
    SET COLOR "black/white"
    SET CURSOR 1,1
    PRINT "            ";             ! erase previous output
    SET CURSOR 1,1
    PRINT "r ="; r
    FOR i = 1 to ntransient           ! x values not plotted
        LET x = f(x,r)
    NEXT i
    SET COLOR "red"
    FOR i = 1 to 0.5*nplot
        LET x = f(x,r)
        ! show different x-values for given value of r
        PLOT r,x
    NEXT i
    ! change color to see if values of x have converged
    SET COLOR "blue"
    FOR i = (0.5*nplot + 1) to nplot
        LET x = f(x,r)
        PLOT r,x
    NEXT i
END SUB

DEF f(x,r) = 4*r*x*(1 - x)
```

Problem 6.2 Qualitative features of the logistic map

a. Use `Program bifurcate` to identify period 2, period 4, and period 8 behavior as in Fig. 6.2. It might be necessary to "zoom in" on a portion of the plot. How many period-doublings can you find?

b. Change the scale so that you can follow the iterations of x from period 4 to period 16 behavior. How does the plot look on this scale in comparison to the original scale?

c. Describe the shape of the trajectory near the bifurcations from period 2 $\rightarrow$ period 4, period 4 $\rightarrow$ 8, etc. These bifurcations are frequently called *pitchfork bifurcations*.

and $x_0 = 0.5$ and compute the trajectory for 200 iterations. Then modify your program so that after each iteration, the operations x = x/10 followed by x = 10 * x are performed. This combination of operations truncates the last digit that your computer retains. A similar effect can be obtained by using the True BASIC `truncate(x,n)` function, which truncates the variable x to n decimal places. Compute the trajectory again and compare your results. Do you find the same discrepancy for $r < r_\infty$?

c. What are the dynamical properties for $r = 0.958$? Can you find other windows of periodic behavior in the interval $r_\infty < r < 1$?

6.3 PERIOD-DOUBLING

The results of the numerical experiments that we did in Section 6.2 have led us to adopt a new vocabulary to describe our observations and probably have convinced you that the dynamical properties of simple deterministic nonlinear systems can be quite complicated.

To gain more insight into how the dynamical behavior depends on r, we introduce a simple graphical method for iterating (6.5). In Fig. 6.3, we show a graph of $f(x)$ versus x for $r = 0.7$. A diagonal line corresponding to $y = x$ intersects the curve $y = f(x)$ at the two fixed points $x^* = 0$ and $x^* = 9/14 \approx 0.642857$. If x_0 is not one of the fixed points, we can find the trajectory in the following manner. Draw a vertical line from $(x = x_0, y = 0)$ to the intersection with the curve $y = f(x)$ at $(x_0, y_0 = f(x_0))$. Next draw a horizontal line from (x_0, y_0) to the intersection with the diagonal line at (y_0, y_0). On this diagonal line $y = x$, and hence the value of x at this intersection is the first iteration $x_1 = y_0$. The second iteration x_2 can be found in the same way. From the point (x_1, y_0), draw a vertical line to the intersection with the curve $y = f(x)$. Keep y fixed at $y = y_1 = f(x_1)$, and draw a horizontal line until it intersects the diagonal line; the value of x at this intersection is x_2. Further iterations can be found by repeating this process.

This graphical method is illustrated in Fig. 6.3 for $r = 0.7$ and $x_0 = 0.9$. If we begin with any x_0 (except $x_0 = 0$ and $x_0 = 1$), continued iterations will converge to the fixed point $x^* \approx 0.642857$. Repeat the procedure shown in Fig. 6.3 by hand and convince yourself that you understand the graphical solution of the iterated values of the map. For this value of r, the fixed point is stable (an attractor of period 1). In contrast, no matter how close x_0 is to the fixed point at $x = 0$, the iterates diverge away from it, and this fixed point is unstable.

How can we explain the qualitative difference between the fixed point at $x = 0$ and $x^* = 0.642857$ for $r = 0.7$? The local slope of the curve $y = f(x)$ determines the distance moved horizontally each time f is iterated. A slope steeper than 45° leads to a value of x further away from its initial value. Hence, the criterion for the stability of a fixed point is that the magnitude of the slope at the fixed point must be less than 45°. That is, if $|df(x)/dx|_{x=x^*}$ is less than unity, then x^* is stable; conversely, if $|df(x)/dx|_{x=x^*}$ is greater than unity, then x^* is unstable. Inspection of $f(x)$ in Fig. 6.3 shows that $x = 0$ is unstable because the slope of $f(x)$ at $x = 0$ is greater than unity. In

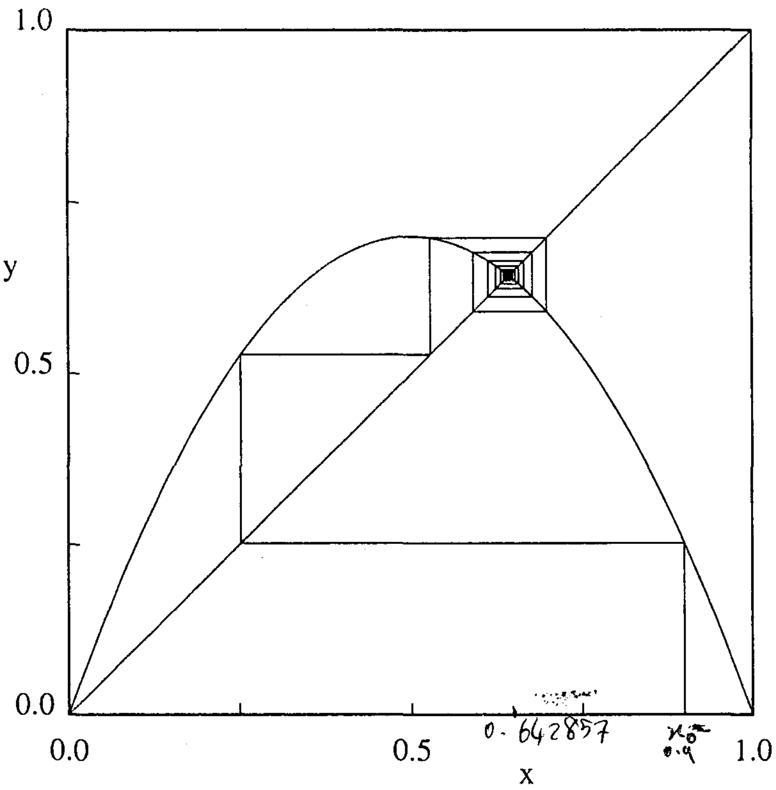


Fig. 6.3 Graphical representation of the iteration of the logistic map (6.5) with $r = 0.7$ and $x_0 = 0.9$. Note that the graphical solution converges to the fixed point $x^* \approx 0.643$.

contrast, the magnitude of the slope of $f(x)$ at $x = x^*$ is less than unity and the fixed point is stable. In Appendix 6A, we use similar analytical arguments to show that

$$x^* = 0 \text{ is stable for } 0 < r < 1/4 \tag{6.6a}$$

and

$$x^* = 1 - \frac{1}{4r} \text{ is stable for } 1/4 < r < 3/4. \tag{6.6b}$$

Thus for $0 < r < 3/4$, the eventual behavior after many iterations is known.

What happens if r is greater than 3/4? From our observations we have found that if r is slightly greater than 3/4, the fixed point of f becomes unstable and gives birth (bifurcates) to a cycle of period 2. Now x returns to the same value only after every second iteration, and the fixed points of $f(f(x))$ are the attractors of $f(x)$. In the following, we adopt the notation $f^{(2)}(x) = f(f(x))$, and write $f^{(n)}(x)$ for the nth iterate of $f(x)$. (Do not confuse $f^{(n)}(x)$ with the nth derivative of $f(x)$.) For example, the second iterate $f^{(2)}(x)$ is given by the fourth-order polynomial:

$$\begin{aligned} f^{(2)}(x) &= 16r^2x(1-x)\big[1-4rx(1-x)\big] \\ &= 16r^2x\big[-4rx^3+8rx^2-(1+4r)x+1\big]. \end{aligned} \tag{6.7}$$

What happens if we increase r still further? Eventually the magnitude of the slope of the fixed points of $f^{(2)}(x)$ exceeds unity and the fixed points of $f^{(2)}(x)$ become unstable. Now the cycle of f is period 4, and we can study the stability of the fixed points of the fourth iterate $f^{(4)}(x) = f^{(2)}\big(f^{(2)}(x)\big) = f\big(f\big(f(f(x))\big)\big)$. These fixed points also eventually bifurcate, and we are led to the phenomena of *period-doubling* as we observed in Problem 6.2.

Program *graph_sol* implements the graphical analysis of $f(x)$. The nth order iterates are defined in DEF `f(x,r,iterate)` using *recursion*. (The quantity `iterate` is 1, 2, and 4 for the functions $f(x)$, $f^{(2)}(x)$, and $f^{(4)}(x)$ respectively.) Recursion is an idea that is simple once you understand it, but it can be difficult to grasp the idea initially. One way to understand how recursion works is to think of a stack, such as a stack of trays in a cafeteria. The first time a recursive function is called, the function is placed on the top of the stack. Each time the function calls itself, an exact copy of the function, with possibly different values of the input parameters, is placed on top of the stack. When a copy of the function is finished, this copy is popped off the top of the stack. To understand the function `f(x,r,iterate)`, suppose we want to compute `f(0.4,0.8,3)`. First we write `f(0.4,0.8,3)` on a piece of paper (see Fig. 6.4a). Follow the statements within the function until another call to `f(0.4,0.8,iterate)` occurs. In this case, the call is to `f(0.4,0.8,iterate-1)` which equals `f(0.4,0.8,2)`. Write `f(0.4,0.8,2)` above `f(0.4,0.8,3)` (see Fig. 6.4b). When you come to the

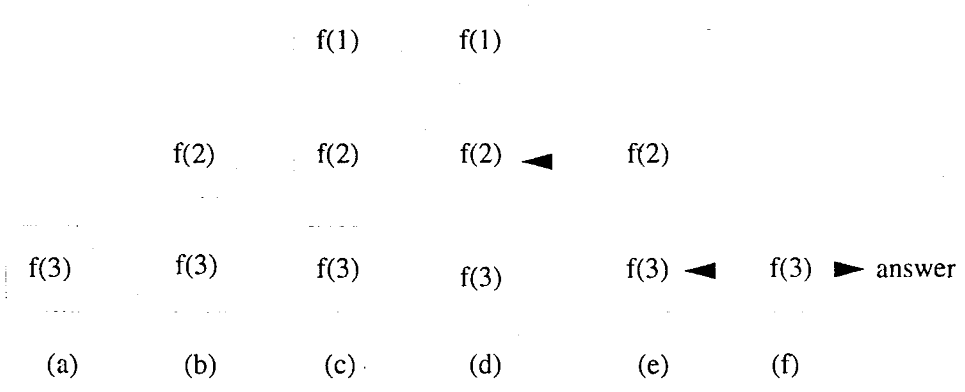


Fig. 6.4 Example of the calculation of `f(0.4,0.8,3)` using the recursive function defined in *Program graph_sol*. The number in each box is the value of the variable `iterate`. The values of $x = 0.4$ and $r = 0.8$ are not shown. The value of $\texttt{f}(x, r, 3) = 0.7842$.

end of the definition of the function, write down the value of `f` that is actually returned, and remove the function from the stack by crossing it out (see Fig. 6.4d). This returned value for `f` equals y if `iterate` > 1, or it is the output of the function for `iterate` = 1. Continue deleting copies of `f` as they are finished, until there are no copies left on the paper. The final value of `f` is the value returned by the computer. Write a miniprogram that defines `f(x,r,iterate)` and prints the value of `f(0.4,0.8,3)`. Is the answer the same as your hand calculation?

```
PROGRAM graph_sol
! graphical solution for trajectory of logistic map
CALL initial(x,r,iterate)
CALL draw_function(r,iterate)
CALL trajectory(x,r,iterate)      ! press any key to stop
END

SUB initial(x0,r,iterate)
    INPUT prompt "control parameter r = ": r
    INPUT prompt "initial value of x = ": x0
    INPUT prompt "iterate of f(x) = ": iterate
    CLEAR
    PRINT "r ="; r
END SUB

SUB draw_function(r,iterate)
    DECLARE DEF f
    LET nplot = 200           ! # of points at which function computed
    LET delta = 1/nplot
    LET margin = 0.1
    SET WINDOW -margin,1 + margin,-margin,1 + margin
    PLOT LINES: 0,0;1,1       ! draw diagonal line y = x
    PLOT LINES: 0,1;0,0;1,0   ! draw axes
    PLOT                      ! left pen
    SET COLOR "red"
    LET x = 0
    FOR i = 0 to nplot
        LET y = f(x,r,iterate)
        PLOT x,y;
        LET x = x + delta
    NEXT i
END SUB

SUB trajectory(x,r,iterate)
    DECLARE DEF f
    LET y0 = 0
    LET x0 = x
    SET COLOR "blue"
    DO
       LET y = f(x,r,iterate)
       PLOT LINES: x0,y0; x0,y; y,y
```

```
      LET x0 = y
      LET y0 = y
      LET x = y
   LOOP until key input
   GET KEY k
END SUB

DEF f(x,r,iterate)              ! f defined by recursive procedure
   IF iterate > 1 then
      LET y = f(x,r,iterate - 1)
      LET f = 4*r*y*(1 - y)
   ELSE
      LET f = 4*r*x*(1 - x)
   END IF
END DEF
```

Problem 6.4 Qualitative properties of the fixed points

a. Use `Program graph_sol` to show graphically that there is a single stable fixed point of $f(x)$ for $r < 3/4$. It would be instructive to insert a pause between each iteration of the map and to show the value of the slope at $y_n = f(x_n)$ in a separate window. At what value of r does the absolute value of this slope exceed unity? Let b_1 denote the value of r at which the fixed point of $f(x)$ bifurcates and becomes unstable. Verify that $b_1 = 0.75$.

b. Describe the trajectory of $f(x)$ for $r = 0.785$. What is the nature of the fixed point given by $x = 1 - 1/4r$? What is the nature of the trajectory if $x_0 = 1 - 1/4r$? What is the period of $f(x)$ for all other choices of x_0? What are the numerical values of the two-point attractor?

c. The function $f(x)$ is symmetrical about $x = \frac{1}{2}$ where $f(x)$ is a maximum. What are the qualitative features of the second iterate $f^{(2)}(x) = f\big(f(x)\big)$ for $r = 0.785$? Is $f^{(2)}(x)$ symmetrical about $x = \frac{1}{2}$? For what value of x does $f^{(2)}(x)$ have a minimum? Iterate $x_{n+1} = f^{(2)}(x_n)$ for $r = 0.785$ and find its two fixed points $x_1{}^*$ and $x_2{}^*$. (Try $x_0 = 0.1$ and $x_0 = 0.3$.) Are the fixed points of $f^{(2)}(x)$ stable or unstable? How do these values of $x_1{}^*$ and $x_2{}^*$ compare with the values of the two-point attractor of $f(x)$? Verify that the slopes of $f^{(2)}(x)$ at $x_1{}^*$ and $x_2{}^*$ are equal.

d. Verify the following properties of the fixed points of $f^{(2)}(x)$. As r is increased, the fixed points of $f^{(2)}(x)$ move apart and the slope of $f^{(2)}(x)$ at the fixed points decreases. What is the value of $r = s_2$ at which one of the two fixed points of $f^{(2)}$ equals $\frac{1}{2}$? What is the value of the other fixed point? What is the slope of $f^{(2)}(x)$ at $x = \frac{1}{2}$? What is the slope at the other fixed point? As r is further increased, the slopes at the fixed points become negative. Finally at $r = b_2 \approx 0.8623$, the slopes at the two fixed points of $f^{(2)}(x)$ equal -1, and the two fixed points of $f^{(2)}$ become unstable. (It can be shown that the exact value of b_2 is $b_2 = (1 + \sqrt{6})/4$.)

e. Show that for r slightly greater than b_2, e.g., $r = 0.87$, there are four stable fixed points of the function $f^{(4)}(x)$. What is the value of $r = s_3$ when one of the fixed points equals $\frac{1}{2}$? What are the values of the three other fixed points at $r = s_3$?

f. Estimate the value of $r = b_3$ at which the four fixed points of $f^{(4)}$ become unstable.

g. Choose $r = s_3$ and estimate the number of iterations that are necessary for the trajectory to converge to period 4 behavior. How does this number of iterations change when neighboring values of r are considered? Choose several values of x_0 so that your results do not depend on the initial conditions.

Problem 6.5 Periodic windows in the chaotic regime

a. If you look closely at the bifurcation diagram in Fig. 6.2, you will see that the region of chaotic behavior for $r > r_\infty$ is interrupted by intervals of periodic behavior. Magnify your bifurcation diagram so that you can look at the interval $0.957107 \leq r \leq 0.960375$, where a periodic trajectory of period 3 occurs. (Period 3 behavior starts at $r = (1 + \sqrt{8})/4$.) What happens to the trajectory for slightly larger r, e.g., for $r = 0.9604$?

b. Plot the map $f^{(3)}(x)$ versus x at $r = 0.96$, a value of r in the period 3 window. Draw the line $y = x$ and determine the intersections with $f^{(3)}(x)$. (Use *Program graph_sol* without calling `SUB trajectory`.) The stable fixed points satisfy the condition $x^* = f^{(3)}(x^*)$. Because $f^{(3)}(x)$ is an eighth-order polynomial, there are eight solutions (including $x = 1$). Find the intersections of $f^{(3)}(x)$ with $y = x$ and identify the three stable fixed points. What are the slopes of $f^{(3)}(x)$ at these points? Then decrease r to $r = 0.957107$, the (approximate) value of r below which the system is chaotic. Draw the line $y = x$ and determine the number of intersections with $f^{(3)}(x)$. Note that at this value of r, the curve $y = f^{(3)}(x)$ is tangent to the diagonal line at the three stable fixed points. For this reason, this type of transition is called a *tangent bifurcation*. Note that there also is an unstable point at $x \approx 0.76$.

c. Plot $x_{n+1} = f^{(3)}(x_n)$ versus n for $r = 0.9571$, a value of r just below the onset of period 3 behavior. How would you describe the behavior of the trajectory? This type of chaotic motion is an example of *intermittency*, that is, nearly periodic behavior interrupted by occasional irregular bursts.

*d. Modify *Program graph_sol* so that you can study the graphical solution of $x_{n+1} = f^{(3)}(x_n)$ for the same value of r as in part (c). That is, "zoom in" on the values of x near the stable fixed points that you found in part (b) for r in the period 3 regime. Note the three narrow channels between the diagonal line $y = x$ and the plot of $f^{(3)}(x)$. The trajectory requires many iterations to squeeze through the channel, and we see period 3 behavior during this time. Eventually, the trajectory escapes from the channel and bounces around until it is sent into a channel at some unpredictable later time.

6.4 UNIVERSAL PROPERTIES AND SELF-SIMILARITY

In Sections 6.2 and 6.3 we found that the trajectory of the logistic map has remarkable properties as a function of the control parameter r. In particular, we found a sequence of period-doublings accumulating to a chaotic trajectory of infinite period at $r = r_\infty$. For most values of $r > r_\infty$, we saw that the trajectory is very sensitive to the initial conditions. We also found "windows" of period 3, 6, 12, ... embedded in the broad regions of chaotic behavior. How typical is this type of behavior? In the following, we will find further numerical evidence that the general behavior of the logistic map is independent of the details of the form (6.5) of $f(x)$.

You might have noticed that the range of r between successive bifurcations becomes smaller as the period increases (see Table 6.1). For example, $b_2 - b_1 = 0.112398$, $b_3 - b_2 = 0.023624$, and $b_4 - b_3 = 0.00508$. A good guess is that the decrease in $b_k - b_{k-1}$ is geometric, i.e., the ratio $(b_k - b_{k-1})/(b_{k+1} - b_k)$ is a constant. You can check that this ratio is not exactly constant, but converges to a constant with increasing k. This behavior suggests that the sequence of values of b_k has a limit and follows a geometrical progression:

$$b_k \approx r_\infty - \text{constant}\ \delta^{-k}, \tag{6.8}$$

where δ is known as the Feigenbaum number. From (6.8) it is easy to show that δ is given by the ratio

$$\delta = \lim_{k\to\infty} \frac{b_k - b_{k-1}}{b_{k+1} - b_k}. \tag{6.9}$$

Problem 6.6 Estimation of the Feigenbaum constant

a. Plot $\delta_k = (b_k - b_{k-1})/(b_{k+1} - b_k)$ versus k using the values of b_k in Table 6.1 and estimate the value of δ. Are the number of decimal places given in

k	b_k
1	0.750 000
2	0.862 372
3	0.886 023
4	0.891 102
5	0.892 190
6	0.892 423
7	0.892 473
8	0.892 484

Table 6.1 Values of the control parameter b_k for the onset of the kth bifurcation. Six decimal places are shown.

Table 6.1 for b_k sufficient for all the values of k shown? The best estimate of δ is

$$\delta = 4.669\,201\,609\,102\,991\ldots \tag{6.10}$$

The number of decimal places in (6.10) is shown to indicate that δ is known precisely. Use (6.8) and (6.10) and the values of b_k to estimate the value of r_∞.

b. In Problem 6.4 we found that one of the four fixed points of $f^{(4)}(x)$ is at $x^* = \frac{1}{2}$ for $r = s_3 \approx 0.87464$. We also found that the convergence to the fixed points of $f^{(4)}(x)$ is more rapid than at nearby values of r. In Appendix 6A we show that these *superstable* trajectories occur whenever one of the fixed points is at $x = \frac{1}{2}$. The values of $r = s_m$ that give superstable trajectories of period 2^{m-1} are much better defined than the points of bifurcation, $r = b_k$. The rapid convergence to the final trajectories also gives better numerical estimates, and we always know one member of the trajectory, namely $x = \frac{1}{2}$. It is reasonable that δ can be defined as in (6.9) with b_k replaced by s_m. Use the values of $s_1 = 0.5$, $s_2 \approx 0.809017$, and $s_3 = 0.874640$ to estimate δ. The numerical values of s_m are found in Project 6.1 by solving the equation $f^{(m)}(x = \frac{1}{2}) = \frac{1}{2}$ numerically; the first eight values of s_m are listed in Table 6.2.

We can associate another number with the series of pitchfork bifurcations. From Fig. 6.3 and Fig. 6.5 we see that each pitchfork bifurcation gives birth to "twins" with the new generation more densely packed than the previous generation. One measure of this density is the maximum distance M_k between the values of x describing the bifurcation (see Fig. 6.5). The disadvantage of using b_k is that the transient behavior of the trajectory is very long at the boundary between two different periodic behaviors. A more convenient measure of the density is the quantity $d_k = x_k{}^* - \frac{1}{2}$, where $x_k{}^*$ is the value of the fixed point nearest to the fixed point $x^* = \frac{1}{2}$. The first two values of d_k are shown in Fig. 6.6 with $d_1 \approx 0.3090$ and $d_2 \approx -0.1164$. The next value is $d_3 \approx 0.0460$. Note that the fixed point nearest to $x = \frac{1}{2}$ alternates from one side of $x = \frac{1}{2}$ to the other. We define the quantity α by the ratio

$$\alpha = \lim_{k\to\infty} -\left(\frac{d_k}{d_{k+1}}\right). \tag{6.11}$$

The estimates $\alpha = 0.3090/0.1164 = 2.65$ for $k = 1$ and $\alpha = 0.1164/0.0460 = 2.53$ for $k = 2$ are consistent with the asymptotic limit $\alpha = 2.5029078750958928485\ldots$

We now give qualitative arguments that suggest that the general behavior of the logistic map in the period-doubling regime is independent of the detailed form of $f(x)$. As we have seen, period-doubling is characterized by self-similarities, e.g., the period-doublings look similar except for a change of scale. We can demonstrate these similarities by comparing $f(x)$ for $r = s_1 = 0.5$ for the superstable trajectory with period 1 to the function $f^{(2)}(x)$ for $r = s_2 \approx 0.809017$ for the superstable trajectory of period 2 (see Fig. 6.7). The function $f(x, r = s_1)$ has unstable fixed points at $x = 0$ and $x = 1$ and a stable fixed point at $x = \frac{1}{2}$. Similarly the function $f^{(2)}(x, r = s_2)$ has a stable fixed point at $x = \frac{1}{2}$ and an unstable fixed point at $x \approx 0.69098$. Note the similar shape,

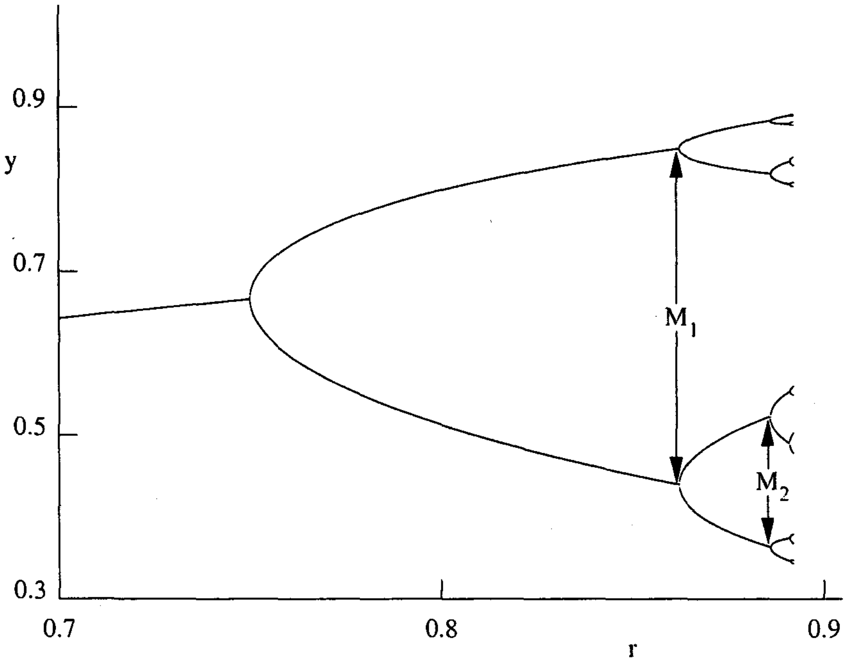


Fig. 6.5 The first few bifurcations of the logistic equation showing the scaling of the maximum distance M_k between the asymptotic values of x describing the bifurcation.

but different scale of the curves in the square box in part (a) and part (b) of Fig. 6.7. This similarity is an example of scaling. That is, if we scale $f^{(2)}$ and change *(renormalize)* the value of r, we can compare $f^{(2)}$ to f. (See Chapter 13 for a discussion of scaling and renormalization in another context.)

Our graphical comparison is meant only to be suggestive. A precise approach shows that if we continue the comparison of the higher-order iterates, e.g., $f^{(4)}(x)$ to $f^{(2)}(x)$, etc., the superposition of functions converges to a universal function that is independent of the form of the original function $f(x)$.

Problem 6.7 Further estimates of the exponents α and δ

a. Write a subroutine to find the appropriate scaling factor and superimpose f and the rescaled form of $f^{(2)}$ found in Fig. 6.7.

b. Use arguments similar to those discussed in the text in Fig. 6.7 and compare the behavior of $f^{(4)}(x, r = s_3)$ in the square about $x = \frac{1}{2}$ with $f^{(2)}(x, r = s_2)$ in its square about $x = \frac{1}{2}$. The size of the squares are determined by the unstable fixed point nearest to $x = \frac{1}{2}$. Find the appropriate scaling factor and superimpose $f^{(2)}$ and the rescaled form of $f^{(4)}$.

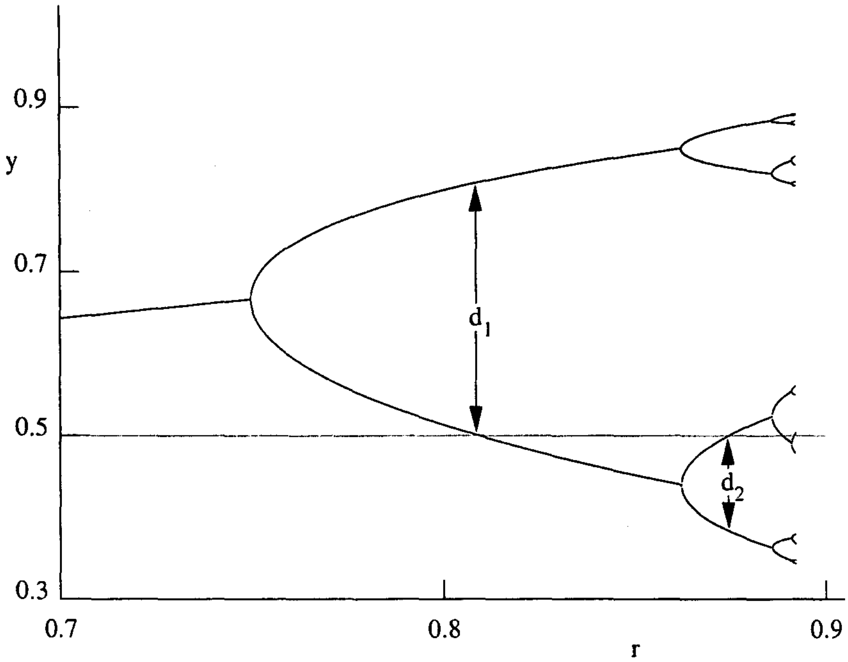


Fig. 6.6 The quantity d_k is the distance from $x^* = 1/2$ to the nearest element of the attractor of period 2^k. It is convenient to use this quantity to determine the exponent α.

It is easy to modify your programs to consider other one-dimensional maps. In Problem 6.8 we consider several one-dimensional maps and determine if they also exhibit the period-doubling route to chaos.

* Problem 6.8 Other one-dimensional maps

Determine the qualitative properties of the one-dimensional maps:

$$f(x) = xe^{r(1-x)} \tag{6.12}$$

$$f(x) = r \sin \pi x. \tag{6.13}$$

The map in (6.12) has been used by ecologists (cf. May) to study a population that is limited at high densities by the effect of epidemic disease. Although it is more complicated than (6.5), its advantage is that the population remains positive no matter what (positive) value is taken for the initial population. There are no restrictions on the maximum value of r, but if r becomes sufficiently large, x eventually becomes effectively zero, rendering the population extinct. What is the behavior of the time series of (6.12) for $r = 1.5$, 2, and 2.7? Describe the qualitative behavior of $f(x)$. Does it have a maximum?

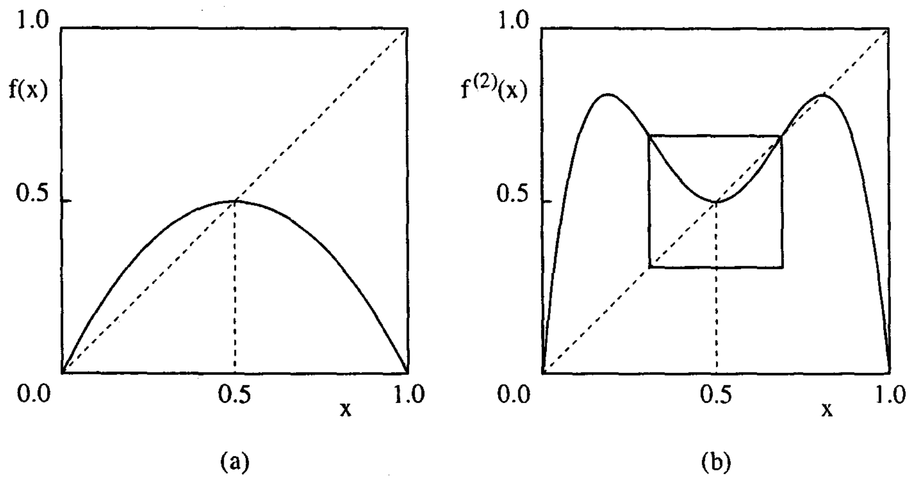


Fig. 6.7 Comparison of $f(x, r)$ for $r = s_1$ with the second iterate $f^{(2)}(x)$ for $r = s_2$. (a) The function $f(x, r = s_1)$ has unstable fixed points at $x = 0$ and $x = 1$ and a stable fixed point at $x = \frac{1}{2}$. (b) The function $f^{(2)}(x, r = s_1)$ has a stable fixed point at $x = \frac{1}{2}$. The unstable fixed point of $f^{(2)}(x)$ nearest to $x = \frac{1}{2}$ occurs at $x \approx 0.69098$, where the curve $f^{(2)}(x)$ intersects the line $y = x$. The upper right-hand corner of the square box in (b) is located at this point, and the center of the box is at $(\frac{1}{2}, \frac{1}{2})$. Note that if we reflect this square about the point $(\frac{1}{2}, \frac{1}{2})$, the shape of the reflected graph in the square box is nearly the same as it is in part (a), but on a smaller scale.

The sine map (6.13) with $0 < r \leq 1$ and $0 \leq x \leq 1$ has no special significance, except that it is nonlinear. If time permits, estimate the value of δ for both maps. What limits the accuracy of your determination of δ?

The above qualitative arguments and numerical results suggest that the quantities α and δ are *universal*, that is, independent of the detailed form of $f(x)$. In contrast, the values of the accumulation point r_∞ and the constant in (6.8) depend on the detailed form of $f(x)$. Feigenbaum has shown that the period-doubling route to chaos and the values of δ and α are universal property of maps that have a quadratic maximum, i.e., $f'(x)|_{x=x_m} = 0$ and $f''(x)|_{x=x_m} < 0$.

Why is the universality of period-doubling and the numbers δ and α more than a curiosity? The reason is that because this behavior is independent of the details, there might exist realistic systems whose underlying dynamics yield the same behavior as the logistic map. Of course, most physical systems are described by differential rather than difference equations. Can these systems exhibit period-doubling behavior? Several workers (cf. Testa et al.) have constructed nonlinear RLC circuits driven by an oscillatory source voltage. The output voltage shows bifurcations, and the measured values of the exponents δ and α are consistent with the predictions of the logistic map.

Of more general interest is the nature of turbulence in fluid systems. Consider a stream of water flowing past several obstacles. We know that at low flow speeds, the water flows past obstacles in a regular and time-independent fashion, called *laminar* flow. As the flow speed is increased (as measured by a dimensionless parameter called the Reynolds number), some swirls develop, but the motion is still time-independent. As the flow speed is increased still further, the swirls break away and start moving downstream. The flow pattern as viewed from the bank becomes time-dependent. For still larger flow speeds, the flow pattern becomes very complex and looks random. We say that the flow pattern has made a transition from laminar flow to *turbulent* flow.

This qualitative description of the transition to chaos in fluid systems is superficially similar to the description of the logistic map. Can fluid systems be analyzed in terms of the simple models of the type we have discussed here? In a few instances such as turbulent convection in a heated saucepan, period doubling and other types of transitions to turbulence have been observed. The type of theory and analysis we have discussed has suggested new concepts and approaches, and the study of turbulent flows is a subject of much current research.

6.5 MEASURING CHAOS

How do we know if a system is chaotic? The most important characteristic of chaos is *sensitivity to initial conditions.* In Problem 6.3 for example, we found that the trajectories starting from $x_0 = 0.5$ and $x_0 = 0.5001$ for $r = 0.91$ become very different after a small number of iterations. Because computers only store floating numbers to a certain number of digits, the implication of this result is that our numerical predictions of the trajectories are restricted to small time intervals. That is, sensitivity to initial conditions implies that even though the logistic map is deterministic, our ability to make numerical predictions is limited.

How can we quantify this lack of predictably? In general, if we start two identical dynamical systems from different initial conditions, we expect that the difference between the trajectories will change as a function of n. In Fig. 6.8 we show a plot of the difference $|\Delta x_n|$ versus n for the same conditions as in Problem 6.3a. We see that roughly speaking, $\ln |\Delta x_n|$ is a linearly increasing function of n. This result indicates that the separation between the trajectories grows exponentially if the system is chaotic. This divergence of the trajectories can be described by the *Lyapunov* exponent, which is defined by the relation:

$$|\Delta x_n| = |\Delta x_0|\, e^{\lambda n}, \tag{6.14}$$

where Δx_n is the difference between the trajectories at time n. If the Lyapunov exponent λ is positive, then nearby trajectories diverge exponentially. Chaotic behavior is characterized by exponential divergence of nearby trajectories.

A naive way of measuring the Lyapunov exponent λ is to run the same dynamical system twice with slightly different initial conditions and measure the difference of the trajectories as a function of n. We used this method to generate Fig. 6.8. Because the rate of separation of the trajectories might depend on the choice of x_0, a better method

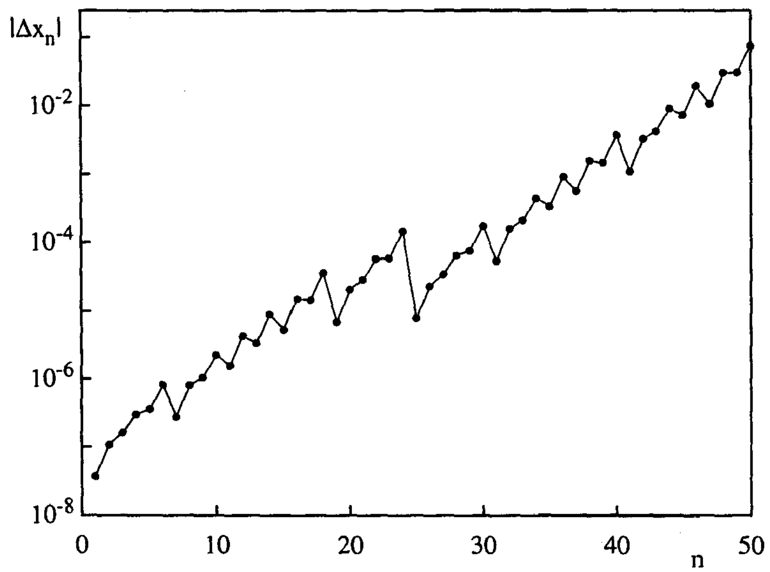


Fig. 6.8 The evolution of the difference Δx_n between the trajectories of the logistic map at $r = 0.91$ for $x_0 = 0.5$ and $x_0 = 0.5001$. The separation between the two trajectories increases with n, the number of iterations, if n is not too large. (Note that $|\Delta x_1| \sim 10^{-8}$ and that the trend is not monotonic.)

would be to compute the rate of separation for many values of x_0. This method would be tedious, because we would have to fit the separation to (6.14) for each value of x_0 and then determine an average value of λ.

A more important limitation of the naive method is that because the trajectory is restricted to the unit interval, the separation $|\Delta x_n|$ ceases to increase when n becomes sufficiently large. However, to make the computation of λ as accurate as possible, we would like to average over as many iterations as possible. Fortunately, there is a better procedure. To understand the procedure, we take the natural logarithm of both sides of (6.14) and write λ as

$$\lambda = \frac{1}{n} \ln \left| \frac{\Delta x_n}{\Delta x_0} \right|. \tag{6.15}$$

Because we want to use the data from the entire trajectory after the transient behavior has ended, we use the fact that

$$\frac{\Delta x_n}{\Delta x_0} = \frac{\Delta x_1}{\Delta x_0} \frac{\Delta x_2}{\Delta x_1} \cdots \frac{\Delta x_n}{\Delta x_{n-1}}. \tag{6.16}$$

Hence, we can express λ as

$$\lambda = \frac{1}{n}\sum_{i=0}^{n-1} \ln \left|\frac{\Delta x_{i+1}}{\Delta x_i}\right|. \tag{6.17}$$

The form (6.17) implies that we can consider x_i for any i as the initial condition.

We see from (6.17) that the problem of computing λ has been reduced to finding the ratio $\Delta x_{i+1}/\Delta x_i$. Because we want to make the initial difference between the two trajectories as small as possible, we are interested in the limit $\Delta x_i \to 0$. The idea of the more sophisticated procedure is to compute the differential dx_i from the equation of motion at the same time that the equation of motion is being iterated. We use the logistic map as an example. The differential of (6.5) can be written as

$$\frac{dx_{i+1}}{dx_i} = f'(x_i) = 4r(1-2x_i). \tag{6.18}$$

We can consider x_i for any i as the initial condition and the ratio dx_{i+1}/dx_i as a measure of the rate of change of x_i. Hence, we can iterate the logistic map as before and use the values of x_i and the relation (6.18) to compute dx_{i+1}/dx_i at each iteration. The Lyapunov exponent is given by

$$\lambda = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \ln \left|f'(x_i)\right|, \tag{6.19}$$

where we begin the sum in (6.19) after the transient behavior is completed. We have included explicitly the limit $n \to \infty$ in (6.19) to remind ourselves to choose n sufficiently large. Note that this procedure weights the points on the attractor correctly, that is, if a particular region of the attractor is not visited often by the trajectory, it does not contribute much to the sum in (6.19).

Problem 6.9 Lyapunov exponent for the logistic map

a. Compute the Lyapunov exponent λ for the logistic map using the naive approach. Choose $r = 0.91$, $x_0 = 0.5$, and $\Delta x_0 = 10^{-6}$, and plot $\ln|\Delta x_n/\Delta x_0|$ versus n. What happens to $\ln|\Delta x_n/\Delta x_0|$ for large n? Estimate λ for $r = 0.91$, $r = 0.97$, and $r = 1.0$. Does your estimate of λ for each value of r depend significantly on your choice of x_0 or Δx_0?

b. Compute λ using the algorithm discussed in the text for $r = 0.76$ to $r = 1.0$ in steps of $\Delta r = 0.01$. What is the sign of λ if the system is not chaotic? Plot λ versus r, and explain your results in terms of behavior of the bifurcation diagram shown in Fig. 6.2. Compare your results for λ with those shown in Fig. 6.9. How does the sign of λ correlate with the behavior of the system as seen in the bifurcation diagram? If $\lambda < 0$, then the two trajectories converge and the system is not chaotic. If $\lambda = 0$, then the trajectories diverge algebraically, i.e., as a power of n. For what value of r is λ a maximum?

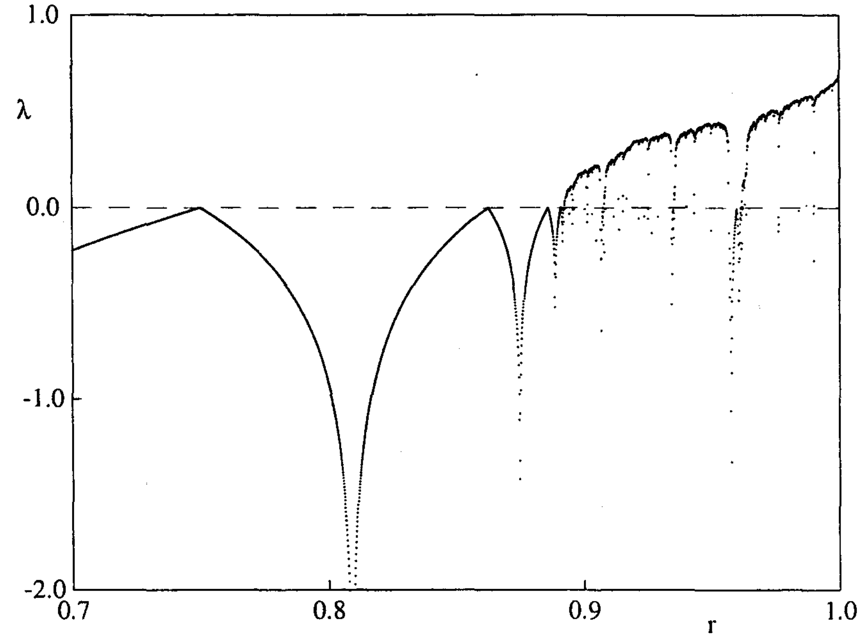


Fig. 6.9 The Lyapunov exponent calculated using the method in (6.19) as a function of the control parameter r. Compare the behavior of λ to the bifurcation diagram in Fig. 6.2. Note that $\lambda < 0$ for $r < 3/4$ and approaches zero at a period doubling bifurcation. A negative spike corresponds to a superstable trajectory. The onset of chaos is visible near $r = 0.892$, where λ first becomes positive. For $r > 0.892$, λ generally increases except for dips below zero whenever a periodic window occurs. Note the large dip due to the period 3 window near $r = 0.96$. For each value of r, the first 1000 iterations were discarded, and 10^5 values of $\ln|f'(x_n)|$ were used to determine λ.

c. In Problem 6.3b we saw that roundoff errors in the chaotic regime make the computation of individual trajectories meaningless. That is, if the system's behavior is chaotic, then small roundoff errors are amplified exponentially in time, and the actual numbers we compute for the trajectory starting from a given initial value are not "real." Given this limitation, how meaningful is our computation of the Lyapunov exponent? Repeat your calculation of λ for $r = 1$ by changing the roundoff error as you did in Problem 6.3b. Does your computed value of λ change? We will encounter a similar question in Chapter 8 where we compute the trajectories of a system of many particles. The answer appears to be that although the trajectory we compute is not the one we thought we were trying to compute, the computed trajectory is close to a possible trajectory of the system. Quantities such as λ that are averaged