No.7

Dr. M.R. Asharif

Simulation

1 - Expectation :

$$\mathcal{E}[X] = \int_{-\infty}^{+\infty} x f(x)\, dx \quad \text{for Continuous random variables.}$$

$$\mathcal{E}[X] = \sum_i x_i \Pr(X = x_i) \quad \text{for Discrete case}$$

2 - Variance : $\mathrm{Var}(X) = \mathcal{E}[(X - \mathcal{E}(X))^2]$

3 - Covariance : $\mathrm{Cov}(X,Y) = \mathcal{E}[(X - \mathcal{E}[X])(Y - \mathcal{E}[Y])]$

4 - Correlation : $\mathrm{Corr}(X,Y) = \dfrac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X) \cdot \mathrm{Var}(Y)}}$

5 - Geometric Dis: $\Pr[X = i] = q^{i-1} p$

$\mathcal{E}[X] = \frac{1}{p}$, $\mathrm{Var}(X) = \frac{q}{p^2}$

6 - Binomial Dis: $B(n,p)$ $\Pr[X = i] = \binom{n}{i} p^i q^{n-i}$

$\mathcal{E}[X] = np$, $\mathrm{Var}(X) = npq$

7 - Poisson Dis : $\Pr[X = i] = \dfrac{e^{-\lambda} \lambda^i}{i!}$

$\mathcal{E}[X] = \lambda$, $\mathrm{Var}[X] = \lambda$

8 - Uniform Dis : $U(\alpha,\beta)$ $\Pr(\alpha \leq X \leq \beta) = \beta - \alpha$

$f(x) = \frac{1}{\beta - \alpha}$, $\underset{U(0,1)}{\mathcal{E}[X]} = \frac{1}{2}$, $\underset{U(0,1)}{\mathrm{Var}(X)} = \frac{1}{12}$

9 - Normal Dis : $N(\mu, \sigma^2)$ $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$

10 - Central Limit theorems: $\lim_{n \to \infty} \Pr\left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right) \leq x \right\} = \Phi(x)$ cdf of $N(0,1)$

Simulation

Q&A No. 8

M.R. Asharif

1 – Exponential probability density function:

$$f(x) = \lambda e^{-\lambda x} \qquad \mathcal{E}[X] = \frac{1}{\lambda} \quad , \quad Var(X) = \frac{1}{\lambda^2}$$

2 – Gamma pdf:

$$f(x) = \frac{e^{-\lambda x} \lambda^{n} x^{n-1}}{\Gamma(n)} \qquad \mathcal{E}[X] = \frac{n}{\lambda} \quad , \quad Var(X) = \frac{n}{\lambda^2}$$

3 – In $S = \sum_{i=1}^{n} X_i$ if X_i are independent random variable with exponential distribution then S has Gamma pdf.

4 – Chi-square pdf:

$$f(x) = \frac{e^{-\frac{x}{2}} x^{\nu/2 - 1}}{\Gamma(\frac{\nu}{2}) 2^{\nu/2}} \qquad \mathcal{E}[X] = \nu \quad , \quad Var(X) = 2\nu$$

5 – Laplace pdf: $f(x) = \frac{\lambda}{2} e^{-\lambda|x|} \qquad \mathcal{E}[X] = 0 \quad , \quad Var(X) = \frac{2}{\lambda^2}$

6 – Logistic pdf: $f(x) = \frac{e^{-x}}{(1+e^{-x})^2} \qquad \mathcal{E}[X] = 0 \quad , \quad Var(X) = \frac{\pi^2}{3}$

7 – Cauchy pdf: $f(x) = \frac{1}{\pi(1+x^2)}$

8 – Beta pdf: $f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}\,\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \qquad 0<x<1$

$$\mathcal{E}[X] = \frac{\alpha}{\alpha+\beta} \quad , \quad Var[X] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Simulation   Q&A No. 9   M.R. Asharif

1- New random variables for old:

If $y = g(x)$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = f_X(g^{-1}(y))\left|\frac{1}{g'(x)}\right|$$

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{1}{g'(g^{-1}(y))}\right|$$

2- In two dimensional case: $w = g(x,y)$, $z = h(x,y)$

Jacobian: $$J = \begin{vmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{vmatrix}$$

$$J^{-1} = \begin{vmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial z} \end{vmatrix}$$

$$f_{W,Z}(w,z) = f_{X,Y}(x,y)\cdot\left|J^{-1}\right|$$

$$f_W(w) = \int f_{W,Z}(w,z)\,dz$$

- See Example 2.4 (Page 35)

- Convolutions: If: $S = X_1 + X_2$

Then: $f_S(s) = f_{X_1}(x) * f_{X_2}(x)$

$$f_S(s) = \int f_{X_1}(x)\cdot f_{X_2}(s-x)\,dx$$

- See Exercises Page 46 ~ 50

Simulation | Q&A No.10 | M.R. Asharif

1- If X, Y are independent, then:

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

See Examples 4.9, 4.10, 4.11 (Page 274~276)

2- Mean or expected value of X_i:

$$\mu_i = \sum_{j=1}^{\infty} x_j P_{X_i}(x_j) = E[X_i] \quad , X_i \text{ is discrete}$$

$$\mu_i = \int_{-\infty}^{+\infty} x f_{X_i}(x)\, dx = E[X_i] \quad , X_i \text{ is Continuous}$$

3- Variance of X_i:

$$\sigma_i^2 = E[(X_i - \mu_i)^2] = E[X_i^2] - \mu_i^2$$

$$Var(X) \geqslant 0 \quad , \quad Var(cX) = c^2 Var(X)$$

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i)$$

4- Covariance shows measure of dependence between two random variables.

$$\text{Covariance} = C_{ij} = E[(X_i - \mu_i)(X_j - \mu_j)]$$

$$C_{ij} = E[X_i X_j] - \mu_i \mu_j$$

$$C_{ii} = \sigma_i^2$$

5- If X_i, X_j are independent, then $C_{ij} = 0$

Simulation   Q&A No. 11   M.R. Asharif

1- If $C_{ij} > 0$ then variables are positively correlated.
If $C_{ij} < 0$ " " " negatively correlated.

2- Correlation: $\rho_{ij} = \dfrac{C_{ij}}{\sqrt{\sigma_i^2 \sigma_j^2}}$

3- Covariance stationary:

$$\mu_i = \mu$$
$$\sigma_i^2 = \sigma^2$$
$$C_{i,i+j} = Cov(X_i, X_{i+j}) = C_{k,k+j}$$

i.e. $C_{i,i+j}$ is independent of i and only on the separation samples j (lag)

or: $\rho_j = \dfrac{C_j}{\sigma^2}$

4- Estimation of means, variances and correlations

$$\bar{X}(n) = \frac{\sum_{i=1}^{n} X_i}{n} \quad , \quad E[\bar{X}(n)] = \mu$$

$$S^2(n) = \frac{\sum_{i=1}^{n} [X_i - \bar{X}(n)]^2}{n-1} \quad , \quad E[S^2(n)] = \sigma^2$$

$$Var[\bar{X}(n)] = \frac{\sigma^2}{n}$$

Simulation

NO.12

Generating Uniform Random Variables

M.R. Asharif

1- Congruential pseudo-random number generators:

$$x_{n+1} = a x_n + b \pmod{m}$$

gives a sequence between $[0, m-1]$

for $x_0 = 89$, $a = 1573$, $b = 19$, $m = 1000$

$x_1 = 16$, $x_2 = 187$

2- $$x_{n+1} = 5 x_n \pmod{11}$$

$x_0 = 1$, $x_1 = 5$, $x_2 = 3$, $x_3 = 4$, $x_4 = 9$

$x_5 = 1$

3- $$x_{n+1} = 5 x_n \pmod{1000}$$

$x_1 = 5$, $x_2 = 25$, $x_3 = 125$, $x_4 = 625$, $x_5 = 125$

Simulation

NO.13

Particular Methods for Non-Uniform Random variables

M.R. Asharif

1- for $U_i(0,1)$
$$N = \sum_{i=1}^{2} U_i$$ has triangular distribution.

2- for $U_i(0,1)$ $N = \sum_{i=1}^{3} U_i$ has bell-shaped distribution.

3- for $U_i(0,1)$ $N = \sum_{i=1}^{12} U_i - 6$

is an approximately normal random variable with:
$$E(N) = 0 \ , \quad Var[N] = 1$$

4- The Box-Muller method:

If: $U_1 , U_2 \longrightarrow U(0,1)$ (uniform)

Then:
$$\begin{cases} N_1 = (-2\log_e U_1)^{1/2} \cos(2\pi U_2) \\ N_2 = (-2\log_e U_1)^{1/2} \sin(2\pi U_2) \end{cases}$$

$N_1 , N_2 \longrightarrow N(0,1)$ (normal distribution)

5- The Polar Marsaglia Method:

If: $V_1 , V_2 \longrightarrow U(-1,1)$ (uniform dis.)

Then:
$$\begin{cases} N_1 = V_2 \left(\dfrac{-2\log W}{W} \right)^{1/2} \\ N_2 = V_1 \left(\dfrac{-2\log W}{W} \right)^{1/2} \end{cases} \qquad (W = V_1^2 + V_2^2)$$

$N_1 , N_2 \longrightarrow N(0,1)$ (Normal Dis.)

(V_1, V_2) are rejected proportion $1 - \frac{\pi}{4}$

1- If $U(0,1)$ is uniform, Then $X = -\log_e U$ has exponential p.d.f.

2- If Y_i's are independent random variables with $\lambda e^{-\lambda x}$ distribution then:

$$G = \sum_{i=1}^{n} Y_i$$

has a gamma $\Gamma(n,\lambda)$ distribution.

3- To simulate $\Gamma(n,\lambda)$ from U_i

$$G = -\frac{1}{\lambda} \sum_{i=1}^{n} \log_e U_i$$

or: $$G = -\frac{1}{\lambda} \log_e \left(\prod_{i=1}^{n} U_i \right)$$

4- Generating Binomial $B(n,p)$ random variables:

∴ Generate $U_1, U_2, \cdots, U_n$ $(U(0,1))$

∴ If $U_i \leq P$ set $B_i = 1$

$U_i \geq P$ set $B_i = 0$

∴ Find $X = \sum_{i=1}^{n} B_i$

X has $B(n,p)$ distribution

5- Poisson variates:

∴ E_i has $\lambda e^{-\lambda x}$ (exponential distribution)

∴ $S_K = \sum_{i=1}^{K} E_i$ (S_K has $\Gamma(K,\lambda)$ distribution)

∴ If $S_K \leq 1 < S_{K+1}$ Then K has a Poisson distribution.

Simulation

No.15

M.R. Asharif

General Methods for Non-Uniform Random Variables

1- Table-look-up method for discrete random variables:

∴ Generate $U(0,1)$

∴ If $0 \leq U < P_0$ set $X=0$

∴ If $\sum_{i=0}^{j-1} P_i \leq U < \sum_{i=0}^{j} P_i$ set $X=j$

2- Generate random variable X with geometric random variables:

$$P_i = Pr[X=i] = (1-P)^{i-1} P$$

∴ Generate $U(0,1)$

Find $X = 1 + \left\lfloor \frac{\log_e U}{\log_e (1-P)} \right\rfloor$

$\lfloor y \rfloor$ is integral part of y.

3- Generate X, where X has poisson distribution given in table bellow:

i	0	1	2	3	4	5
$Pr(X \leq i)$	0.1353	0.4060	0.6767	0.8571	0.9473	0.9834

From a uniform random number in table below:

U	0.0318	0.5321	0.2489	0.9583	0.5678	0.6777
X	0	2	1	5	2	3

1 - Simulate an exponential density random variable X where $f(x) = \lambda e^{-\lambda x}$

from a $U(0,1)$ by using inversion method:

$$\because \quad F(x) = \int_0^x f(x)\,dx = 1 - e^{-\lambda x}$$

$$X = F^{-1}(U)$$

$$U = F(X) = 1 - e^{-\lambda X}$$

$$X = -\frac{1}{\lambda} \log_e (1-U)$$

or:

$$X = -\frac{1}{\lambda} \log_e U$$