with my solution
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Simulation Exam **Name:**
3-rd year undergraduate No:
2007-2-5 Last Term Examination
Time: 90 minutes (write answers in boxes)

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1- The Fibonacci sequence is defined as follows:

$Fib(1)=Fib(2)=1$

$Fib(n)=Fib(n-1)+Fib(n-2) \qquad for\ n>=3$

It can be shown that:

$Fib(n)=\{[(1+\sqrt{5})/2]^n-[(1-\sqrt{5})/2]^n\}/\sqrt{5}$

Find Fib(10), both by direct method and using the above equation. *10%*

fib(3) = 2, 3, 5, 8, 13, 21, 34, 55

$fib(10)=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{10}-\left(\frac{1-\sqrt{5}}{2}\right)^{10}\right]=\boxed{55}$ fib(10)

$\boxed{Fib(10) = 55}$

2- Simulate the Gamma distributed random variables, G, with $\Gamma(n,\lambda)$ for $n=3$, $\lambda = 0.5$ from the following uniform distributed random variables, $U(0,1)$:
U1=0.95, U2=0.23, U3=0.60, *10%*

(Hint: See page 82)

$G=-\frac{1}{\lambda}\log_e \prod_{i=1}^{n} U_i, \quad \Gamma(n,\lambda)=\Gamma(3,0.5)$

$G=-\frac{1}{0.5}\log_e(0.95\times0.23\times0.6)=-2\log_e 0.1311=-2(-2.03)=4.06$

$\boxed{G= 4.06}$

3- In the following chaotic system: *10%*

$x(n+1)=4\,r\,x(n)\,[1-x(n)]$

If $r=0.5$, find the attractor of this chaotic system.

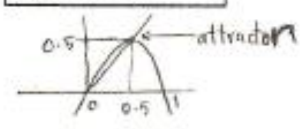
(Hint: See chap. 6, page 136)

$y=2x-2x^2$

$y'=2-4x \Rightarrow 0 \to x=0.5 \to y=0.5$

$y=x \to x=2x-2x^2 \to x=0.5 \to y=0.5$ attractor

$\boxed{x(\infty)= 0.5}$



4- Use the table-look-up method to simulate random variables X from $U(0,1)$.
Where the p.d.f of X is: $f(x)=e^{-x}/(1+e^{-x})^2$, $-\infty<x<\infty$ *10%*
Also, find the value of X when $U=0.5$

(Hint: see page 95-96)

$U=F(x)=\int_{-\infty}^{x} f(x)\,dx=\int_{-\infty}^{x}\frac{e^{-x}}{(1+e^{-x})^2}dx$

$U=F(x)=\left.\frac{1}{1+e^{-x}}\right]_{-\infty}^{x}=\frac{1}{1+e^{-x}}$

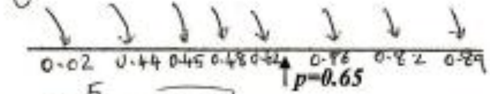
$1+e^{-x}=\frac{1}{U} \to e^{-x}=\frac{1}{U}-1=\frac{1-U}{U} \to x=\boxed{\log_e\frac{U}{1-U}}$ $\quad x\Big|_{U=0.5}=\log_e\frac{0.5}{0.5}=0$

$\boxed{X=\log_e\frac{U}{1-U} \quad X|_{U=0.5}=0}$

5- Simulate a Binomial random variable X with $B(8,0.65)$ from a set of uniform random variables $U(0,1)$, by using Bernouli random variable, where:
U1=0.48, U2=0.89, U3=0.76, U4=0.45, U5=0.02, U6=0.82, U7=0.44, U8=0.62 *10%*

(Hint: See page 82)

$\boxed{X= 5}$



$X=\sum_{i=1}^{5}B_i=\boxed{5}$

6- In randomised response technique (RRT), if we have $p_0=0.4$ for answering *[N]*, *($1-p_0=0.6$)* for answering *[E]* and *Pr[Yes|N]=0.9*, and total probability from survey is: *Pr[Yes]=0.72*, what is the *Pr[Yes| E]*?

(Hint: See page 51)

10%

Pr[Yes | E] = 0.6

$$P_r[\text{yes}] = P_r[\text{Yes}|N]\,p_0 + P_r[\text{Yes}|E]\times(1-p_0)$$

$$0.72 = 0.9\times0.4 + P_r[\text{Yes}|E]\times0.6$$

$$P_r[\text{yes}|E] = \frac{0.72-0.36}{0.6} = \frac{0.36}{0.6} = 0.6$$

7- Simulate the normal distributed random variables *(N1, N2)* by using Polar-Marsaglia method (rejection method) from each pair of the following uniform distributed random variables: *(Hint: See page 80)*
(V1,V2)=(0.1,-0.2) , (V1,V2)=(0.5,0.9), (V1,V2)=(0.8,-0.6)

10%

(N1,N2)= (-2.19, 1.095)	(N1,N2)= rejected	, (N1,N2)= (0, 0)

$$W = V_1^2+V_2^2 = (0.1)^2+(-0.2)^2 = 0.01+0.04 = 0.05$$

$$N_1 = V_2\left(\frac{-2\log_e W}{W}\right)^{1/2} = -0.2\left(\frac{-2\log_e 0.05}{0.05}\right)^{1/2} = -0.2\left(\frac{-2(-3)}{0.05}\right)^{1/2} = -0.2(120)^{1/2} = -0.2\times10.\ldots = -2.19$$

$$N_2 = V_1\left(\frac{-2\log_e W}{W}\right)^{1/2} = 0.1(120)^{1/2} = 0.1\times10.95 = 1.095$$

$$W = 0.25+0.81 = 1.06 > 1 \rightarrow \text{rejected}$$

$$W = 0.64+0.36 = 1$$

$$N_1 = -0.6\left(\frac{-2\log_e 1}{1}\right)^{1/2}$$

$$N_1 = 0$$

$$N_2 = 0$$

8-Simulate random variable *X* with geometric distribution and *p=0.2* from *U(0,1)=0.486*

(Hint: See page 93 Eq. 5.4)

X= 4

10%

$$X = 1+\left\lfloor\frac{\log_e U}{\log_e(1-p)}\right\rfloor = 1+\left\lfloor\frac{\log_e 0.486}{\log_e(1-0.2)}\right\rfloor$$

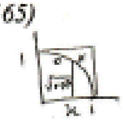
$$X = 1+\left\lfloor\frac{-0.72}{-0.22}\right\rfloor = 1+\lfloor3.27\rfloor = 1+3 = 4$$

9- In estimation of π by integral method. Compare the variances of Hit-or-Miss Monte Carlo and Crude Monte Carlo methods. Which one has the lower variance.

10%

(Hint: See page 162-165)

$$\frac{\pi}{4} = \int_0^1 \sqrt{1-x^2}\,dx$$



Hit-or-miss $\rightarrow$ Var(π)= $\frac{2.697}{n}$

Crude $\rightarrow$ Var(π)= $\frac{0.398}{n}$

Crude Monte Carlo has lower Variance

Hit-or-miss: $\frac{R}{n} = \frac{\pi}{4}$

$$\pi = \frac{4R}{n}$$

$R \rightarrow$ Binomial $B(n, \frac{\pi}{4})$

$$\mathrm{Var}(R) = npq = n\times\frac{\pi}{4}\left(1-\frac{\pi}{4}\right)$$

$$\mathrm{Var}(\pi) = \mathrm{Var}\left(\frac{4R}{n}\right) = \frac{16}{n^2}\mathrm{Var}(R) = \frac{16}{n^2}\times n\times\frac{\pi}{4}\left(1-\frac{\pi}{4}\right) = \frac{\pi(4-\pi)}{n} = \frac{2.697}{n}$$

(See back page)

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Care - Monte Carlo :

$$\frac{\pi}{4} = E\left[\sqrt{1-U^2}\right] = I = \int_0^1 \sqrt{1-x^2}\,dx$$

$$I = \frac{1}{n}\sum_{i=1}^{n}\sqrt{1-U_i^2}$$

$$Var(I) = \frac{1}{n} Var\left(\sqrt{1-U^2}\right) = \frac{1}{n}\left[\mathcal{E}\left(\sqrt{1-U^2}\right)^2 - \mathcal{E}^2\left(\sqrt{1-U^2}\right)\right]$$

$$= \frac{1}{n}\left[\int_0^1 (1-x^2)\,dx - \left(\int_0^1 \sqrt{1-x^2}\,dx\right)^2\right]$$

$$= \frac{1}{n}\left[\left[x - \frac{x^3}{3}\right]_0^1 - \left(\frac{\pi}{4}\right)^2\right]$$

$$= \frac{1}{n}\left[\frac{2}{3} - \frac{\pi^2}{16}\right] = \frac{0.0498}{n}$$

$$Var(\pi) = Var(4I) = 16\,Var(I) = 16\,\frac{0.0498}{2n} = \frac{0.398}{n}$$