3.6 図 3.16 に示す離散時間信号のフーリエ変換を求めよ。

(a)

$$\begin{aligned} X_1(j\omega) &= \exp(j2\omega T) + 2\exp(j\omega T) + 3 + 2\exp(-j\omega T) + \exp(-j2\omega T) \\ &= \{\exp(j\omega T) + 1 + \exp(-j\omega T)\}^2 \end{aligned}$$

$\exp(j\omega T) + 1 + \exp(-j\omega T) = \frac{\sin(3/2)\omega T}{\sin(1/2)\omega T}$なので、

$$X_1(j\omega) = \left(\frac{\sin(3/2)\omega T}{\sin(1/2)\omega T}\right)^2$$

(b)

$$\begin{aligned} X_2(j\omega) &= 1 + 2\exp(-j\omega T) + 3\exp(-j2\omega T) + 2\exp(-j3\omega T) + \exp(-j4\omega T) \\ &= \{\exp(j2\omega T) + 2\exp(j\omega T) + 3 + 2\exp(-j\omega T) + \exp(-j2\omega T)\}\exp(-j2\omega T) \end{aligned}$$

(a) より、

$$X_2(j\omega) = \left(\frac{\sin(3/2)\omega T}{\sin(1/2)\omega T}\right)^2 \exp(-j2\omega T)$$

(c)

$$X_3(j\omega) = 1 + 2\exp(-j2\omega T) + 3\exp(-j4\omega T) + 2\exp(-j6\omega T) + \exp(-j8\omega T)$$

$$\begin{aligned} X_3'(j\omega) &= 0.5\exp(j\omega T) + 1 + 1.5\exp(-j\omega T) + 2\exp(-j2\omega T) + 2.5\exp(-j3\omega T) \\ &\quad +3\exp(-j4\omega T) + 2.5\exp(-j5\omega T) + 2\exp(-j6\omega T) + 1.5\exp(-j7\omega T) \\ &\quad +\exp(-j8\omega T) + 0.5\exp(-j9\omega T) \end{aligned}$$

とすると、

$$X_3(j\omega) = \frac{1}{2}\left[X_3'(j\omega) + X_3'\{j(\omega + (\pi/T)\}\right]$$

[証明]

$$\begin{aligned} X_3'\{j(\omega + \pi/T)\} &= 0.5\exp\{j(\omega + \pi/T)T\} + 1 + 1.5\exp\{-j(\omega + \pi/T)T\} + 2\exp\{-j2(\omega + \pi/T)T\} \\ &\quad +2.5\exp\{-j3(\omega + \pi/T)T\} + 3\exp\{-j4(\omega + \pi/T)T\} + 2.5\exp\{-j5(\omega + \pi/T)T\} \\ &\quad +2\exp\{-j6(\omega + \pi/T)T\} + 1.5\exp\{-j7(\omega + \pi/T)T\} + \exp\{-j8(\omega + \pi/T)T\} \end{aligned}$$

$$
\begin{aligned}
&+0.5\exp\{-j9(\omega+\pi/T)T\}\\
=\ & 0.5\exp(j\pi)\exp(j\omega T)+1+1.5\exp(-j\pi)\exp(-j\omega T)\\
&+2\exp(-j2\pi)\exp(-j2\omega T)+2.5\exp(-j3\pi)\exp(-j3\omega T)\\
&+3\exp(-j4\pi)\exp(-j4\omega T)+2.5\exp(-j5\pi)\exp(-j5\omega T)\\
&+2\exp(-j6\pi)\exp(-j6\omega T)+1.5\exp(-j7\pi)\exp(-j7\omega T)\\
&+\exp(-j8\pi)\exp(-j8\omega T)+0.5\exp(-j9\pi)\exp(-j9\omega T)\\
=\ & -0.5\exp(j\omega T)+1-1.5\exp(-j\omega T)+2\exp(-j2\omega T)-2.5\exp(-j3\omega T)\\
&+3\exp(-j4\omega T)-2.5\exp(-j5\omega T)+2\exp(-j6\omega T)-1.5\exp(-j7\omega T)\\
&\exp(-j8\omega T)-0.5\exp(-j9\omega T)
\end{aligned}
$$

よって、

$$
X_3(j\omega) = \frac{1}{2}\left[X_3'(j\omega)+X_3'\{j(\omega+(\pi/T)\}\right]\ (\text{証明終わり})
$$

$$
\begin{aligned}
X_3'(j\omega) =\ & \frac{1}{2}[\exp(j5\omega T)+2\exp(j4\omega T)+3\exp(j3\omega T)+4\exp(j2\omega T)\\
&+5\exp(j\omega T)+6+5\exp(-j\omega T)+4\exp(-j2\omega T)+3\exp(-j3\omega T)\\
&+2\exp(-j4\omega T)+\exp(-j5\omega T)]\exp(-j4\omega T)\\
=\ & \frac{1}{2}\{\exp(j\frac{5}{2}\omega T)+\exp(j\frac{3}{2}\omega T)+\exp(j\frac{1}{2}\omega T)+\exp(-j\frac{1}{2}\omega T)\\
&+\exp(j\frac{3}{2}\omega T)+\exp(-j\frac{5}{2}\omega T)\}^2\exp(-j4\omega T)
\end{aligned}
$$

ここで、

$$
\begin{aligned}
S =\ & \exp(j\frac{5}{2}\omega T)+\exp(j\frac{3}{2}\omega T)+\exp(j\frac{1}{2}\omega T)+\exp(-j\frac{1}{2}\omega T)\\
&+\exp(j\frac{3}{2}\omega T)+\exp(-j\frac{5}{2}\omega T)
\end{aligned}
$$

とすると、Sは初項 $\exp(j(5/2)\omega T)$、項比 $\exp(-j\omega T)$ の等比数列の和なので、

$$
S=\frac{\exp(j\frac{5}{2}\omega T)-\exp(-j\frac{7}{2}\omega T)}{1-\exp(-j\omega T)}=\frac{\exp(j3\omega T)-\exp(-j3\omega T)}{\exp(j\frac{1}{2}\omega T)-\exp(-j\frac{1}{2}\omega T)}=\frac{\sin 3\omega T}{\sin(1/2)\omega T}
$$

よって、

$$
X_3'(j\omega) = \frac{1}{2}\left(\frac{\sin 3\omega T}{\sin(1/2)\omega T}\right)^2\exp(-j4\omega T)
$$

(1)

$$\sin(2\pi fnT) = \frac{1}{2j}[\exp(j2\pi fnT - \exp(-j2\pi fnT)]$$

$$\begin{aligned} X_4(j\omega) &= \sum_{n=-\infty}^{\infty}[u(nT) - u(nT - 10T)]\frac{1}{2j}[\exp(j2\pi fnT) - \exp(-j2\pi fnT)]\exp(-j\omega nT) \\ &= \frac{1}{2j}\sum_{n=0}^{9}\exp(j2\pi fnT)\exp(-j\omega nT) \textcircled{+} \frac{1}{2j}\sum_{n=0}^{9}\exp(-j2\pi fnT)\exp(-j\omega nT) \\ &= \frac{1}{2j}\sum_{n=0}^{9}\exp(-j(\omega - 2\pi f)nT) \textcircled{+} \frac{1}{2j}\sum_{n=0}^{9}\exp(-j(\omega + 2\pi f)nT) \end{aligned}$$

$\omega - 2\pi f = \omega'$ とすると、

$$\sum_{n=0}^{9}\exp(-j\omega' nT) = \frac{\sin 5\omega' T}{\sin(1/2)\omega' T}\exp(-j(11/2)\omega' T)$$

よって、

$$X_4(j\omega) = \frac{1}{2j}\left[\frac{\sin 5(\omega - 2\pi f)T}{\sin\frac{\omega-2\pi f}{2}T}\exp(-j\frac{11}{2}(\omega - 2\pi f)T) + \frac{\sin 5(\omega + 2\pi f)T}{\sin\frac{\omega+2\pi f}{T}}\exp(-j\frac{11}{2}(\omega + 2\pi f)T)\right]$$

(2) (1) と同様に

$$X_5(j\omega) = \frac{1}{2j}\left[\frac{\sin 10(\omega - 2\pi f)T}{\sin\frac{\omega-2\pi f}{2}T}\exp(-j\frac{21}{2}(\omega - 2\pi f)T) + \frac{\sin 10(\omega + 2\pi f)T}{\sin\frac{\omega+2\pi f}{T}}\exp(-j\frac{21}{2}(\omega - 2\pi f)T)\right]$$