Digital Image Processing:

CHAPTER 5

Image Restoration

Degradation Model

Input Output relationship

$$g(x, y) = H[f(x, y)] + \eta(x, y).$$

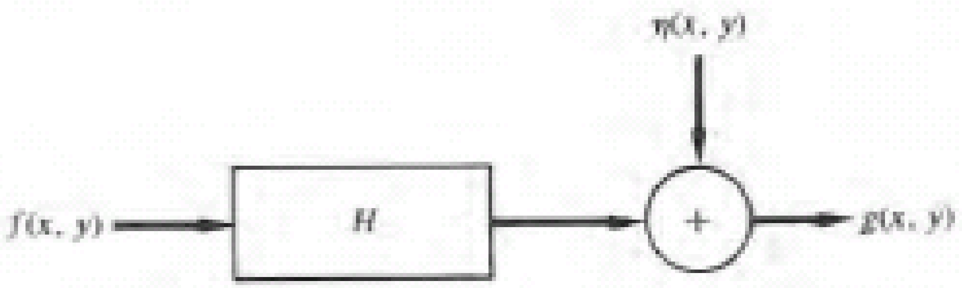

Figure 5.1 *A model of the image degradation process.*

An operator having the input –output relationship

$$H[f(x - \alpha, y - \beta)] = g(x - \alpha, y - \beta)$$

Degradation Model

Degradation Model for Continuous Function

$$f(x, y) = \iint_{-\infty}^{\infty} f(\alpha, \beta)\delta(x - \alpha, y - \beta)\, d\alpha\, d\beta.$$

Since $f(\alpha, \beta)$ is independent of x and y, and from the homogeneity property,

$$g(x, y) = \iint_{-\infty}^{\infty} f(\alpha, \beta) H[\delta(x - \alpha, y - \beta)]\, d\alpha\, d\beta. \qquad (5.1\text{-}9)$$

Impulse response

The term

$$h(x, \alpha, y, \beta) = H[\delta(x - \alpha, y - \beta)]$$

Degradation Model

Discrete Formulation

$$g_e(x) = \sum_{m=0}^{M-1} f_e(m) h_e(x - m)$$

$$\mathbf{f} = \begin{bmatrix} f_e(0) \\ f_e(1) \\ \vdots \\ f_e(M-1) \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} g_e(0) \\ g_e(1) \\ \vdots \\ g_e(M-1) \end{bmatrix}$$

$\mathbf{H}$ is the $M \times M$ matrix

$$\mathbf{H} = \begin{bmatrix} h_e(0) & h_e(-1) & h_e(-2) & \cdots & h_e(-M+1) \\ h_e(1) & h_e(0) & h_e(-1) & \cdots & h_e(-M+2) \\ h_e(2) & h_e(1) & h_e(0) & \cdots & h_e(-M+3) \\ \vdots & & & & \\ h_e(M-1) & h_e(M-2) & h_e(M-3) & \cdots & h_e(0) \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} h(0) & & & & h(2) & h(1) \\ h(1) & h(0) & & & & h(2) \\ h(2) & h(1) & h(0) & & & \\ & h(2) & h(1) & h(0) & & \\ & & h(2) & h(1) & h(0) & \\ & & & h(2) & h(1)) & h(0) \end{bmatrix}$$

Digitalization of Circulant And Block-Circulant Matrices

Circulant Matrices:

$$\mathbf{H} = \begin{bmatrix} h_e(0) & h_e(M-1) & h_e(M-2) & \cdots & h_e(1) \\ h_e(1) & h_e(0) & h_e(M-1) & \cdots & h_e(2) \\ h_e(2) & h_e(1) & h_e(0) & \cdots & h_e(3) \\ \vdots & & & & \\ h_e(M-1) & h_e(M-2) & h_e(M-3) & \cdots & h_e(0) \end{bmatrix}$$

Scalar Functio $\lambda(k)$ and a vector $\mathbf{w}(k)$ as

$$\lambda(k) = h_e(0) + h_e(M-1)\exp\left[j\frac{2\pi}{M}k\right] + h_e(M-2)\exp\left[j\frac{2\pi}{M}2k\right] + \cdots + h_e(1)\exp\left[j\frac{2\pi}{M}(M-1)k\right]$$

Digitalization of Circulant And Block-Circulant Matrices

The transformation matrix for diagonalizing block circulants is constructed as follows. Let

$$w_M(i, m) = \exp\left[j \frac{2\pi}{M} im\right] \quad (5.2\text{-}12)$$

and

$$w_N(k, n) = \exp\left[j \frac{2\pi}{N} kn\right]. \quad (5.2\text{-}13)$$

Based on this notation, we define a matrix $\mathbf{W}$ of size $MN \times MN$ and containing M^2 partitions of size $N \times N$. The imth partition of $\mathbf{W}$ is

$$\mathbf{W}(i, m) = w_M(i, m)\mathbf{W}_N \quad (5.2\text{-}14)$$

for $i, m = 0, 1, 2, \ldots, M - 1$. Then $\mathbf{W}_N$ is an $N \times N$ matrix with elements

$$W_N(k, n) = w_N(k, n) \quad (5.2\text{-}15)$$

for $k, n = 0, 1, 2, \ldots, N - 1$.

The inverse matrix $\mathbf{W}^{-1}$ is also of size $MN \times MN$ with M^2 partitions of size $N \times N$. The imth partition of $\mathbf{W}^{-1}$, symbolized as $\mathbf{W}^{-1}(i, m)$, is

$$\mathbf{W}^{-1}(i, m) = \frac{1}{M} w_M^{-1}(i, m)\mathbf{W}_N^{-1} \quad (5.2\text{-}16)$$

where $w_M^{-1}(i, m)$ is

$$w_M^{-1}(i, m) = \exp\left[-j\frac{2\pi}{M}im\right] \tag{5.2-17}$$

for $i, m = 0, 1, 2, \ldots, M-1$. The matrix $\mathbf{W}_N^{-1}$ has elements

$$W_N^{-1}(k, n) = \frac{1}{N}w_N^{-1}(k, n) \tag{5.2-18}$$

where

$$w_N^{-1}(k, n) = \exp\left[-j\frac{2\pi}{N}kn\right] \tag{5.2-19}$$

for $k, n = 0, 1, 2, \ldots, N-1$. It can be verified by direct substitution of the elements of $\mathbf{W}$ and $\mathbf{W}^{-1}$ that

$$\mathbf{W}\mathbf{W}^{-1} = \mathbf{W}^{-1}\mathbf{W} = \mathbf{I} \tag{5.2-20}$$

where $\mathbf{I}$ is the $MN \times MN$ identity matrix.

From the results in Section 5.2.1, and if $\mathbf{H}$ is a block-circulant matrix, it can be shown (Hunt [1973]) that

$$\mathbf{H} = \mathbf{W}\mathbf{D}\mathbf{W}^{-1} \tag{5.2-21}$$

or

$$\mathbf{D} = \mathbf{W}^{-1}\mathbf{H}\mathbf{W} \tag{5.2-22}$$

where $\mathbf{D}$ is a diagonal matrix whose elements $D(k, k)$ are related to the discrete Fourier transform of the extended function $h_e(x, y)$ discussed in Section 5.1.3. Moreover, the transpose of $\mathbf{H}$, denoted $\mathbf{H}^T$, is

$$\mathbf{H}^T = \mathbf{W}\mathbf{D}^*\mathbf{W}^{-1} \tag{5.2-23}$$

where $\mathbf{D}^*$ is the complex conjugate of $\mathbf{D}$.

Algebraic Approach

Unconstrained Restoration

$$\mathbf{n} = \mathbf{g} - \mathbf{Hf}.$$

Norm OF the Nose:

$$\|\mathbf{n}\|^2 = \|\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}\|^2$$

$$\|\mathbf{n}\|^2 = \mathbf{n}^T\mathbf{n} \quad \text{and} \quad \|\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}\|^2 = (\mathbf{g} - \mathbf{H}\hat{\mathbf{f}})^T(\mathbf{g} - \mathbf{H}\hat{\mathbf{f}})$$

Algebraic Approach

Constrained Restoration:

$$J(\hat{\mathbf{f}}) = \|\mathbf{Q}\hat{\mathbf{f}}\|^2 + \alpha(\|\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}\|^2 - \|\mathbf{n}\|^2)$$

Inverse Filtering

$$\hat{\mathbf{f}} = \mathbf{H}^{-1}\mathbf{g} = (\mathbf{W}\mathbf{D}\mathbf{W}^{-1})^{-1}\mathbf{g} = \mathbf{W}\mathbf{D}^{-1}\mathbf{W}^{-1}\mathbf{g}.$$

Premultiplying both sides

$$\mathbf{W}^{-1}\hat{\mathbf{f}} = \mathbf{D}^{-1}\mathbf{W}^{-1}\mathbf{g}.$$

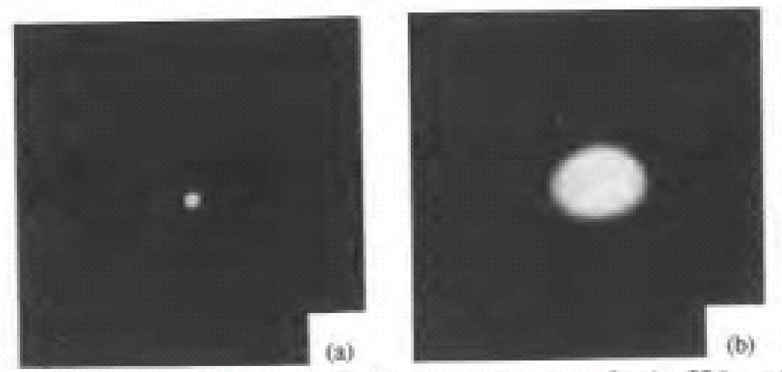

Figure 5.2 Blurring of a point source to obtain $H(u, v)$.

Inverse Filtering

Removal of Blur Caused by Uniform Linear Motion:

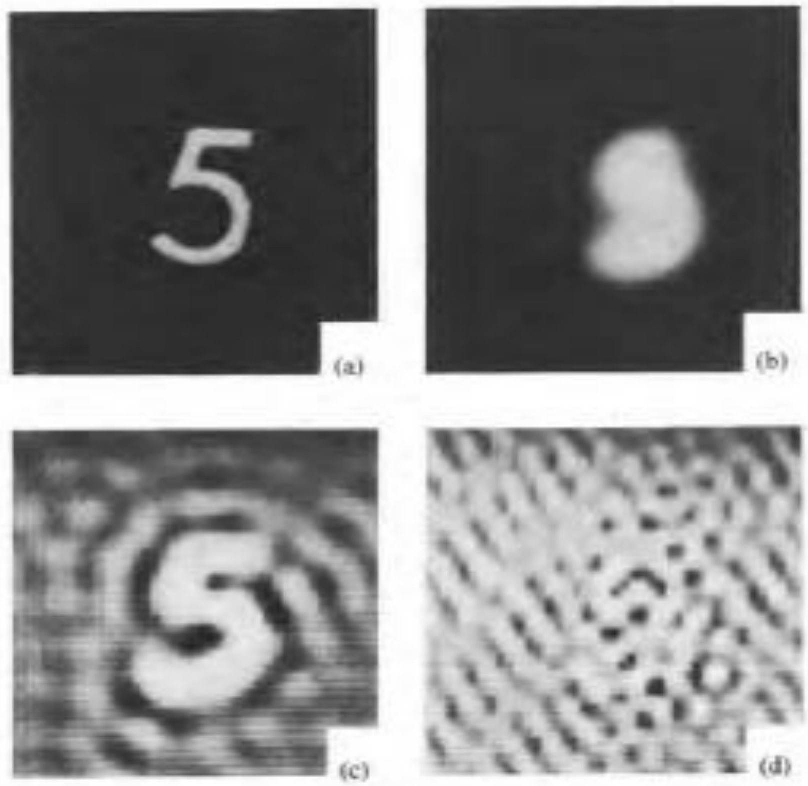

Figure 5.3 Example of image restoration by inverse filtering: (a) original image $f(x, y)$; (b) degraded (blurred) image $g(x, y)$; (c) result of restoration by considering a neighborhood about the origin of the uv plane that does not include excessively small values of $H(u, v)$; (d) result of using a larger neighborhood in which this condition does not hold. (From McGlamery [1967].)

Inverse Filtering

exposure, it follows that

$$g(x, y) = \int_0^T f[x - x_0(t), y - y_0(t)]\, dt \qquad (5.4\text{-}7)$$

where $g(x, y)$ is the blurred image.

From Eq. (3.1-9), the Fourier transform of Eq. (5.4-7) is

$$\begin{aligned} G(u, v) &= \iint_{-\infty}^{\infty} g(x, y) \exp[-j2\pi(ux + vy)]\, dx\, dy \\ &= \iint_{-\infty}^{\infty} \left[\int_0^T f[x - x_0(t), y - y_0(t)]\, dt\right] \exp[-j2\pi(ux + vy)]\, dx\, dy. \end{aligned} \qquad (5.4\text{-}8)$$

Reversing the order of integration allows Eq. (5.4-8) to be expressed in the form

$$G(u, v) = \int_0^T \left[\iint_{-\infty}^{\infty} f[x - x_0(t), y - y_0(t)] \exp[-j2\pi(ux + vy)]\, dx\, dy\right] dt. \qquad (5.4\text{-}9)$$

The term inside the outer brackets is the Fourier transform of the displaced function $f[x - x_0(t), y - y_0(t)]$. Using Eq. (3.3-7b) then yields the relation

$$\begin{aligned} G(u, v) &= \int_0^T F(u, v) \exp\{-j2\pi[ux_0(t) + vy_0(t)]\}\, dt \\ &= F(u, v) \int_0^T \exp\{-j2\pi[ux_0(t) + vy_0(t)]\}\, dt \end{aligned} \qquad (5.4\text{-}10)$$

where the last step follows from the fact that $F(u, v)$ is independent of t.

By defining

$$H(u, v) = \int_0^T \exp\{-j2\pi[ux_0(t) + vy_0(t)]\}\, dt \qquad (5.4\text{-}11)$$

Eq. (5.4-10) may be expressed in the familiar form

$$G(u, v) = H(u, v) F(u, v). \qquad (5.4\text{-}12)$$

If the nature of the motion variables $x_0(t)$ and $y_0(t)$ is known, the transfer function $H(u, v)$ can be obtained directly from Eq. (5.4-11). As an illustration, suppose that the image in question undergoes uniform linear motion in the x direction only, at a rate given by $x_0(t) = at/T$. When $t = T$, the image has

Inverse Filtering

been displaced by a total distance a. With $y_0(t) = 0$, Eq. (5.4-11) yields

$$
\begin{aligned}
H(u, v) &= \int_0^T \exp[-j2\pi u x_0(t)]\, dt \\
&= \int_0^T \exp[-j2\pi uat/T]\, dt \qquad (5.4\text{-}13)\\
&= \frac{T}{\pi ua} \sin(\pi ua) e^{-j\pi ua}.
\end{aligned}
$$

Obviously, H vanishes at values of u given by $u = n/a$, where n is an integer.

When $f(x, y)$ is zero (or known) outside an interval $0 \le x \le L$, the problem presented by Eq. (5.4-13) can be avoided and the image completely reconstructed from a knowledge of $g(x, y)$ in this interval. Because y is time invariant, suppressing this variable temporarily allows Eq. (5.4-7) to be written as

$$
\begin{aligned}
g(x) &= \int_0^T f[x - x_0(t)]\, dt \qquad (5.4\text{-}14)\\
&= \int_0^T f\left(x - \frac{at}{T}\right) dt \qquad 0 \le x \le L.
\end{aligned}
$$

Substituting $\tau = x - at/T$ in this expression and ignoring a scale factor yields

$$g(x) = \int_{x-a}^{x} f(\tau)\, d\tau \qquad 0 \le x \le L. \qquad (5.4\text{-}15)$$

Then, by differentiation with respect to x (using Liebnitz's rule),

$$g'(x) = f(x) - f(x - a) \qquad 0 \le x \le L \qquad (5.4\text{-}16)$$

or

$$f(x) = g'(x) + f(x - a) \qquad 0 \le x \le L. \qquad (5.4\text{-}17)$$

In the following development a convenient assumption is that $L = Ka$, where K is an integer. Then the variable x may be expressed in the form

$$x = z + ma \qquad (5.4\text{-}18)$$

where z takes on values in the interval $[0, a]$ and m is the integral part of (x/a). For example, if $a = 2$ and $x = 3.5$, then $m = 1$ (the integral part of 3.5/2), and $z = 1.5$. Clearly, $z + ma = 3.5$, as required. Note also that, for $L = Ka$, the index m can assume any of the integer values $0, 1, \ldots, K - 1$. For instance, when $x = L$, then $z = a$ and $m = K - 1$.

13:24

Inverse Filtering

Substitution of Eq. (5.4-18) into Eq. (5.4-17) yields

$$f(z + ma) = g'(z + ma) + f[z + (m - 1)a]. \qquad (5.4\text{-}19)$$

Next, denoting $\phi(z)$ as the portion of the scene that moves into the range $0 \leq z < a$ during exposure gives

$$\phi(z) = f(z - a) \qquad 0 \leq z < a. \qquad (5.4\text{-}20)$$

Equation (5.4-19) can be solved recursively in terms of $\phi(z)$. Thus for $m = 0$,

$$\begin{aligned} f(z) &= g'(z) + f(z - a) \\ &= g'(z) + \phi(z). \end{aligned} \qquad (5.4\text{-}21)$$

For $m = 1$, Eq. (5.4-19) becomes

$$f(z + a) = g'(z + a) + f(z). \qquad (5.4\text{-}22)$$

Substituting Eq. (5.4-21) into Eq. (5.4-22) yields

$$f(z + a) = g'(z + a) + g'(z) + \phi(z). \qquad (5.4\text{-}23)$$

In the next step, letting $m = 2$ results in the expression

$$f(z + 2a) = g'(z + 2a) + f(z + a) \qquad (5.4\text{-}24)$$

or, substituting Eq. (5.4-23) for $f(z + a)$,

$$f(z + 2a) = g'(z + 2a) + g'(z + a) + g'(z) + \phi(z). \qquad (5.4\text{-}25)$$

Continuing with this procedure finally yields

$$f(z + ma) = \sum_{k=0}^{m} g'(z + ka) + \phi(z). \qquad (5.4\text{-}26)$$

However, as $x = z + ma$, Eq. (5.4-26) may be expressed in the form

$$f(x) = \sum_{k=0}^{m} g'(x - ka) + \phi(x - ma) \qquad 0 \leq x \leq L. \qquad (5.4\text{-}27)$$

Because $g(x)$ is known, the problem is reduced to that of estimating $\phi(x)$.

One way to estimate this function directly from the blurred image is as follows. First note that, as x varies from 0 to L, m ranges from 0 to $K - 1$. The argument of ϕ is $(x - ma)$, which is always in the range $0 \leq x - ma <$

Inverse Filtering

From Eqs. (5.4-28) and (5.4-29), we have the final result:

$$f(x) = A - \frac{1}{K}\sum_{k=0}^{K-1}\sum_{j=0}^{k} g'[x - ma + (k - j)a] + \sum_{j=0}^{m} g'(x - ja) \qquad (5.4\text{-}35)$$

for $0 \leq x \leq L$. Reintroducing the suppressed variable y yields

$$f(x, y) = A - \frac{1}{K}\sum_{k=0}^{K-1}\sum_{j=0}^{k} g'[x - ma + (k - j)a, y] + \sum_{j=0}^{m} g'(x - ja, y) \qquad (5.4\text{-}36)$$

for $0 \leq x, y \leq L$. As before, $f(x, y)$ is assumed to be a square image. Interchanging x and y in the right-hand side of Eq. (5.4-36) would give the reconstruction of an image that moves only in the y direction during exposure. The concepts presented can also be used to derive a deblurring expression that takes into account simultaneous uniform motion in both directions.

Example: The image shown in Fig. 5.4(a) was blurred by uniform linear motion in one direction during exposure, with the total distance traveled being approximately equal to ⅛ the width of the photograph. Figure 5.4(b) shows the deblurred result obtained by using Eq. (5.4-36) with x and y interchanged because motion is in the y direction. The error in the approximation given by this equation is not objectionable. ❑

(a) (b)

Figure 5.4 *(a) Image blurred by uniform linear motion; (b) image restored by using Eq. (5.4-36). (From Sondhi [1972].)*

Least Mean Square (Wiener) Filter

Let $\mathbf{R}_f$ and $\mathbf{R}_n$ be the correlation matrices of $\mathbf{f}$ and $\mathbf{n}$, defined respectively by the equations

$$\mathbf{R}_f = E\{\mathbf{f}\mathbf{f}^T\} \tag{5.5-1}$$

and

$$\mathbf{R}_n = E\{\mathbf{n}\mathbf{n}^T\} \tag{5.5-2}$$

where $E\{\cdot\}$ denotes the expected value operation, and $\mathbf{f}$ and $\mathbf{n}$ are as defined in Section 5.1.3. The ijth element of $\mathbf{R}_f$ is given by $E\{f_i f_j\}$, which is the correlation between the ith and the jth elements of $\mathbf{f}$. Similarly, the ijth element of $\mathbf{R}_n$ gives the correlation between the two corresponding elements in $\mathbf{n}$. Since the elements of $\mathbf{f}$ and $\mathbf{n}$ are real, $E\{f_i f_j\} = E\{f_j f_i\}$, $E\{n_i n_j\} = E\{n_j n_i\}$, and it follows that $\mathbf{R}_f$ and $\mathbf{R}_n$ are real symmetric matrices. For most image functions the correlation between pixels (that is, elements of $\mathbf{f}$ or $\mathbf{n}$) does not extend beyond a distance of 20 to 30 pixels in the image, so a typical correlation matrix has a band of nonzero elements about the main diagonal and zeros in the right upper and left lower corner regions. Based on the assumption that the correlation between any two pixels is a function of the distance between the pixels and not their position, $\mathbf{R}_f$ and $\mathbf{R}_n$ can be made to approximate block-circulant matrices and therefore can be diagonalized by the matrix $\mathbf{W}$ with the procedure described in Section 5.2.2 (Andrews and Hunt [1977]). Using $\mathbf{A}$ and $\mathbf{B}$ to denote matrices gives

$$\mathbf{R}_f = \mathbf{W}\mathbf{A}\mathbf{W}^{-1} \tag{5.5-3}$$

and

$$\mathbf{R}_n = \mathbf{W}\mathbf{B}\mathbf{W}^{-1}. \tag{5.5-4}$$

Just as the elements of the diagonal matrix $\mathbf{D}$ in the relation $\mathbf{H} = \mathbf{W}\mathbf{D}\mathbf{W}^{-1}$ correspond to the Fourier transform of the block elements of $\mathbf{H}$, the elements of $\mathbf{A}$ and $\mathbf{B}$ are the transforms of the correlation elements in $\mathbf{R}_f$ and $\mathbf{R}_n$, respectively. As indicated in Problem 3.4, the Fourier transform of these correlations is called the *power spectrum* (or *spectral density*) of $f_e(x, y)$ and $\eta_e(x, y)$, respectively and is denoted $S_f(u, v)$ and $S_\eta(u, v)$ in the following discussion.

Defining

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{R}_f^{-1}\mathbf{R}_n \tag{5.5-5}$$

13:24

Wiener Filter

and substituting this expression in Eq. (5.3-9) gives

$$\hat{\mathbf{f}} = (\mathbf{H}^T\mathbf{H} + \gamma \mathbf{R}_f^{-1}\mathbf{R}_n)^{-1}\mathbf{H}^T\mathbf{g}. \qquad (5.5\text{-}6)$$

Using Eqs. (5.2-21), (5.2-23), (5.5-3), and (5.5-4) yields

$$\hat{\mathbf{f}} = (\mathbf{W}\mathbf{D}^*\mathbf{D}\mathbf{W}^{-1} + \gamma \mathbf{W}\mathbf{A}^{-1}\mathbf{B}\mathbf{W}^{-1})^{-1}\mathbf{W}\mathbf{D}^*\mathbf{W}^{-1}\mathbf{g}. \qquad (5.5\text{-}7)$$

Multiplying both sides by $\mathbf{W}^{-1}$ and performing some matrix manipulations reduces Eq. (5.5-7) to

$$\mathbf{W}^{-1}\hat{\mathbf{f}} = (\mathbf{D}^*\mathbf{D} + \gamma \mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{D}^*\mathbf{W}^{-1}\mathbf{g}. \qquad (5.5\text{-}8)$$

Keeping in mind the meaning of the elements of **A** and **B**, recognizing that the matrices inside the parentheses are diagonal, and making use of the concepts developed in Section 5.2.3, allows writing the elements of Eq. (5.5-8) in the form

$$\begin{aligned}\hat{F}(u, v) &= \left[\frac{H^*(u, v)}{|H(u, v)|^2 + \gamma[S_n(u, v)/S_f(u, v)]}\right]G(u, v) \\ &= \left[\frac{1}{H(u, v)}\frac{|H(u, v)|^2}{|H(u, v)|^2 + \gamma[S_n(u, v)/S_f(u, v)]}\right]G(u, v)\end{aligned} \qquad (5.5\text{-}9)$$

for $u, v = 0, 1, 2, \ldots, N - 1$, where $|H(u, v)|^2 = H^*(u, v)H(u, v)$ and it is assumed that $M = N$.

When $\gamma = 1$, the term inside the outer brackets in Eq. (5.5-9) reduces to the so-called *Wiener filter*. If γ is variable this expression is called the *parametric Wiener filter*. In the absence of noise, $S_n(u, v) = 0$ and the Wiener filter reduces to the ideal inverse filter discussed in Section 5.4. However, when $\gamma = 1$, the use of Eq. (5.5-9) no longer yields an optimal solution in the sense defined in Section 5.3.2 because, as pointed out in that section, γ must be adjusted to satisfy the constraint $\|\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}\|^2 = \|\mathbf{n}\|^2$. It can be shown however, that the solution obtained with $\gamma = 1$ *is* optimal in the sense that it minimizes the quantity $E\{[f(x, y) - \hat{f}(x, y)]^2\}$. Clearly, this is a statistical criterion that treats f and $\hat{f}$ as random variables.

When $S_n(u, v)$ and $S_f(u, v)$ are unknown (a problem often encountered in practice) approximating Eq. (5.5-9) by the relation

$$\hat{F}(u, v) = \left[\frac{1}{H(u, v)}\frac{|H(u, v)|^2}{|H(u, v)|^2 + K}\right]G(u, v) \qquad (5.5\text{-}10)$$

Wiener Filter

where K is a constant, sometimes is useful. An example of results obtained with (5.5-10) follows. The problem of selecting the optimal γ for image restoration is discussed in some detail in Section 5.6.

Example: The first column in Fig. 5.5 shows three pictures of a domino corrupted by linear motion (at $-45°$ with respect to the horizontal) and noise whose variance at any point in the image was proportional to the brightness of the point. The three images were generated by varying the constant of proportionality so that the ratios of maximum brightness to noise amplitude were 1, 10, and 100, respectively, as shown on the left in Fig. 5.5. The Fourier spectra of the degraded images are shown in Fig. 5.5(b).

Since the effects of uniform linear motion can be expressed analytically, an equation describing $H(u, v)$ can be obtained without difficulty, as shown in Section 5.4.2. Figure 5.5(c) was obtained by direct inverse filtering following the procedure described in Section 5.4.1. The results are dominated by noise, but as the third image shows, the inverse filter successfully removed the degradation (blur) caused by motion. By contrast, Fig. 5.5(d) shows the results

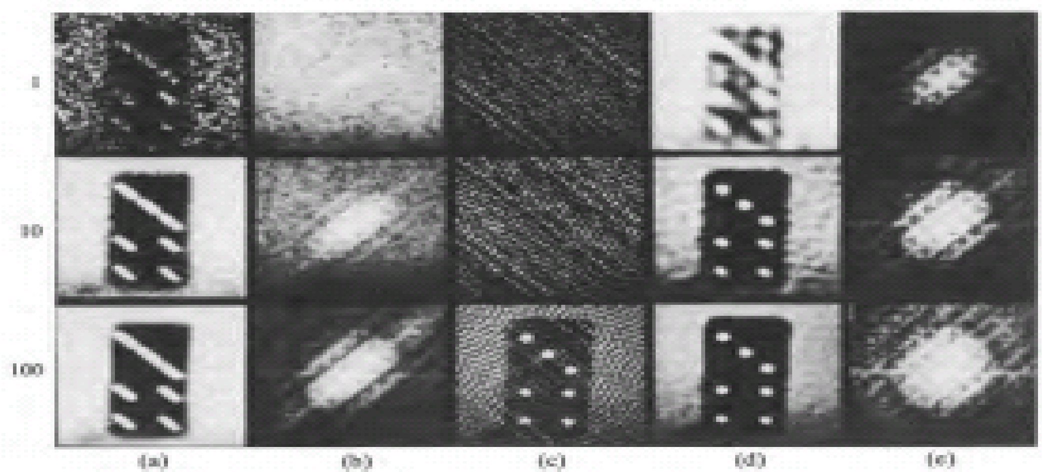

Figure 5.5 Example of image restoration by inverse and Wiener filters: (a) degraded images and (b) their Fourier spectra; (c) images restored by inverse filtering; (d) images restored by Wiener filtering; (e) Fourier spectra of images in (d). (From Harris [1968].)

Constrained Lest Squares Reoration

$$\frac{\partial^2 f(x)}{\partial x^2} \approx f(x+1) - 2f(x) + f(x-1).$$

A criterion based on this expression, then, might be to minimize $(\partial^2 f/\partial x^2)^2$ over x; that is,

$$\text{minimize}\left[\sum_x [f(x+1) - 2f(x) + f(x-1)]^2\right] \qquad (5.6\text{-}2)$$

or, in matrix notation,

$$\text{minimize}\,(\mathbf{f}^T\mathbf{C}^T\mathbf{C}\mathbf{f}) \qquad (5.6\text{-}3)$$

Constrained Lest Squares Reoration

where

$$\mathbf{C} = \begin{bmatrix} 1 & & & & & & \\ -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & & \ddots & & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & \\ & & & & & 1 & \end{bmatrix} \qquad (5.6\text{-}4)$$

is a "smoothing" matrix, and $\mathbf{f}$ is a vector whose elements are the samples of $f(x)$.

In the 2-D case we consider a direct extension of Eq. (5.6-1). In this case the criterion is to

$$\text{minimize}\left[\frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2}\right]^2 \qquad (5.6\text{-}5)$$

where the derivative function is approximated by the expression

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= [2f(x, y) - f(x + 1, y) - f(x - 1, y)] \\ &\quad + [2f(x, y) - f(x, y + 1) - f(x, y - 1)] \\ &= 4f(x, y) - [f(x + 1, y) + f(x - 1, y) + f(x, y + 1) \\ &\quad + f(x, y - 1)]. \end{aligned} \qquad (5.6\text{-}6)$$

The derivative function given in Eq. (5.6-5) is the Laplacian operator discussed in Section 3.3.7.

Equation (5.6-6) can be implemented directly in a computer. However, the same operation can be carried out by convolving $f(x, y)$ with the operator

$$p(x, y) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} \qquad (5.6\text{-}7)$$

As indicated in Section 5.1.3, wraparound error in the discrete convolution process is avoided by extending $f(x, y)$ and $p(x, y)$. Having already considered

13:24

Constrained Lest Squares Reoration

the formation of $f_e(x, y)$, we form $p_e(x, y)$ in the same manner:

$$p_e(x, y) = \begin{cases} p(x, y) & 0 \leq x \leq 2 \quad \text{and} \quad 0 \leq y \leq 2 \\ 0 & 3 \leq x \leq M - 1 \quad \text{or} \quad 3 \leq y \leq N - 1. \end{cases}$$

If $f(x, y)$ is of size $A \times B$, we choose $M \geq A + 3 - 1$ and $N \geq B + 3 - 1$, because $p(x, y)$ is of size 3×3.

The convolution of the extended functions then is

$$g_e(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_e(m, n) p_e(x - m, y - n) \tag{5.6-8}$$

which agrees with Eq. (5.1-23).

Following an argument similar to the one given in Section 5.1.3 allows expression of the smoothness criterion in matrix form. First, we construct a block-circulant matrix of the form

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_{M-1} & \mathbf{C}_{M-2} & \cdots & \mathbf{C}_1 \\ \mathbf{C}_1 & \mathbf{C}_0 & \mathbf{C}_{M-1} & \cdots & \mathbf{C}_2 \\ \mathbf{C}_2 & \mathbf{C}_1 & \mathbf{C}_0 & \cdots & \mathbf{C}_3 \\ \vdots & & & & \\ \mathbf{C}_{M-1} & \mathbf{C}_{M-2} & \mathbf{C}_{M-3} & \cdots & \mathbf{C}_0 \end{bmatrix} \tag{5.6-9}$$

where each submatrix $\mathbf{C}_j$ is an $N \times N$ circulant constructed from the jth row of $p_e(x, y)$; that is,

$$\mathbf{C}_j = \begin{bmatrix} p_e(j, 0) & p_e(j, N-1) & \cdots & p_e(j, 1) \\ p_e(j, 1) & p_e(j, 0) & \cdots & p_e(j, 2) \\ \vdots & & & \\ p_e(j, N-1) & p_e(j, N-2) & \cdots & p_e(j, 0) \end{bmatrix} \tag{5.6-10}$$

Since $\mathbf{C}$ is block circulant, it is diagonalized by the matrix $\mathbf{W}$ defined in Section 5.2.2. In other words,

$$\mathbf{E} = \mathbf{W}^{-1}\mathbf{C}\mathbf{W} \tag{5.6-11}$$

Constrained Lest Squares Reoration

where $\mathbf{E}$ is a diagonal matrix whose elements are given by

$$E(k, i) = \begin{cases} P\left(\left[\frac{k}{N}\right], k \bmod N\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases} \tag{5.6-12}$$

as in Eq. (5.2-39). In this case $P(u, v)$ is the 2-D Fourier transform of $p_e(x, y)$. As with Eqs. (5.2-37) and (5.2-39), the assumption is that Eq. (5.6-12) has been scaled by the factor MN.

The convolution operation described above is equivalent to implementing Eq. (5.6-6), so the smoothness criterion of Eq. (5.6-5) takes the same form as Eq. (5.6-3):

$$\text{minimize}\{\mathbf{f}^T\mathbf{C}^T\mathbf{C}\mathbf{f}\} \tag{5.6-13}$$

where $\mathbf{f}$ is an MN-dimensional vector and $\mathbf{C}$ is of size $MN \times MN$. By letting $\mathbf{Q} = \mathbf{C}$, and recalling that $\|\mathbf{Qf}\|^2 = (\mathbf{Qf})^T(\mathbf{Qf}) = \mathbf{f}^T\mathbf{Q}^T\mathbf{Qf}$, this criterion may be expressed as

$$\text{minimize}\|\mathbf{Qf}\|^2 \tag{5.6-14}$$

which is the same form used in Section 5.3.2. In fact, if we require that the constraint $\|\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}\|^2 = \|\mathbf{n}\|^2$ be satisfied, the optimal solution is given by Eq. (5.3-9) with $\mathbf{Q} = \mathbf{C}$:

$$\hat{\mathbf{f}} = (\mathbf{H}^T\mathbf{H} + \gamma\mathbf{C}^T\mathbf{C})^{-1}\mathbf{H}^T\mathbf{g}. \tag{5.6-15}$$

Using Eqs. (5.2-21), (5.2-23), and (5.6-11), allows Eq. (5.6-15) to be expressed as

$$\hat{\mathbf{f}} = (\mathbf{W}\mathbf{D}^*\mathbf{D}\mathbf{W}^{-1} + \gamma\mathbf{W}\mathbf{E}^*\mathbf{E}\mathbf{W}^{-1})^{-1}\mathbf{W}\mathbf{D}^*\mathbf{W}^{-1}\mathbf{g}. \tag{5.6-16}$$

Multiplying both sides by $\mathbf{W}^{-1}$ and performing some matrix manipulations reduces Eq. (5.6-16) to

$$\mathbf{W}^{-1}\hat{\mathbf{f}} = (\mathbf{D}^*\mathbf{D} + \gamma\mathbf{E}^*\mathbf{E})^{-1}\mathbf{D}^*\mathbf{W}^{-1}\mathbf{g}. \tag{5.6-17}$$

Keeping in mind that the elements inside the parentheses are diagonal and making use of the concepts developed in Section 5.2.3, allows expressing the elements of Eq. (5.6-17) in the form

$$\hat{F}(u, v) = \left[\frac{H^*(u, v)}{|H(u, v)|^2 + \gamma|P(u, v)|^2}\right]G(u, v) \tag{5.6-18}$$

Constrained Lest Squares Reoration

for $u, v = 0, 1, 2, \ldots, N-1$, where $|H(u,v)|^2 = H^*(u,v)H(u,v)$, and we have assumed that $M = N$. Note that Eq. (5.6-18) resembles the parametric Wiener filter derived in Section 5.5. The principal difference between Eqs. (5.5-9) and (5.6-18) is that the latter does not require explicit knowledge of statistical parameters other than an estimate of the noise mean and variance.

The general formulation given in Eq. (5.3-9) requires that γ be adjusted to satisfy the constraint $\|\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}\|^2 = \|\mathbf{n}\|^2$. Thus the solution given in Eq. (5.6-18) can be optimal only when γ satisfies this condition. An iterative procedure for estimating this parameter follows.

Define a residual vector $\mathbf{r}$ as

$$\mathbf{r} = \mathbf{g} - \mathbf{H}\hat{\mathbf{f}}. \tag{5.6-19}$$

Substituting Eq. (5.6-15) for $\hat{\mathbf{f}}$ yields

$$\mathbf{r} = \mathbf{g} - \mathbf{H}(\mathbf{H}^T\mathbf{H} + \gamma\mathbf{C}^T\mathbf{C})^{-1}\mathbf{H}^T\mathbf{g}. \tag{5.6-20}$$

Equation (5.6.20) indicates that $\mathbf{r}$ is a function of γ. In fact, it can be shown (Hunt [1973]) that

$$\begin{aligned}\phi(\gamma) &= \mathbf{r}^T\mathbf{r} \\ &= \|\mathbf{r}\|^2\end{aligned} \tag{5.6-21}$$

is a monotonically increasing function of γ. What we want to do is adjust γ so that

$$\|\mathbf{r}\|^2 = \|\mathbf{n}\|^2 \pm a, \tag{5.6-22}$$

where a is an accuracy factor. Clearly, if $\|\mathbf{r}\|^2 = \|\mathbf{n}\|^2$ the constraint $\|\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}\|^2 = \|\mathbf{n}\|^2$ will be strictly satisfied, in view of Eq. (5.6-19).

Because $\phi(\gamma)$ is monotonic, finding a γ that satisfies Eq. (5.6-17) is not difficult. One simple approach is to

(1) specify an initial value of γ;

(2) compute $\hat{\mathbf{f}}$ and $\|\mathbf{r}\|^2$; and

(3) stop if Eq. (5.6-22) is satisfied; otherwise return to step 2 after increasing γ if $\|\mathbf{r}\|^2 < \|\mathbf{n}\|^2 - a$ or decreasing γ if $\|\mathbf{r}\|^2 > \|\mathbf{n}\|^2 + a$.

Other procedures such as a Newton–Raphson algorithm can be used to improve speed of convergence.

Implementation of these concepts requires some knowledge about $\|\mathbf{n}\|^2$. The variance of $\eta_e(x, y)$ is

$$\begin{aligned}\sigma_e^2 &= E\{[\eta_e(x,y) - \bar{\eta}_e]^2\} \\ &= E[\eta_e^2(x,y)] - \bar{\eta}_e^2\end{aligned} \tag{5.6-23}$$

Constrained Lest Squares Reoration

where

$$\bar{\eta}_e = \frac{1}{(M-1)(N-1)} \sum_x \sum_y \eta_e(x, y) \qquad (5.6\text{-}24)$$

is the mean value of $\eta_e(x, y)$. If a sample average is used to approximate the expected value of $\eta_e^2(x, y)$, Eq. (5.6-23) becomes

$$\sigma_e^2 = \frac{1}{(M-1)(N-1)} \sum_x \sum_y \eta_e^2(x, y) - \bar{\eta}_e^2. \qquad (5.6\text{-}25)$$

The summation term simply indicates squaring and adding all values in the array $\eta_e(x, y)$, $x = 0, 1, 2, \ldots, M-1$, and $y = 0, 1, 2, \ldots, N-1$. This manipulation is simply the product $\mathbf{n}^T\mathbf{n}$, which, by definition, equals $\|\mathbf{n}\|^2$. Thus Eq. (5.6-25) reduces to

$$\sigma_e^2 = \frac{\|\mathbf{n}\|^2}{(M-1)(N-1)} - \bar{\eta}_e^2 \qquad (5.6\text{-}26)$$

or

$$\|\mathbf{n}\|^2 = (M-1)(N-1)[\sigma_e^2 + \bar{\eta}_e^2]. \qquad (5.6\text{-}27)$$

The importance of this equation is that it allows determination of a value for the constraint in terms of the noise mean and variance, quantities that, if not known, can often be approximated or measured in practice.

The constrained least squares restoration procedure can be summarized as follows.

Step 1. Choose an initial value of γ and obtain an estimate of $\|\mathbf{n}\|^2$ by using Eq. (5.6-27).

Step 2. Compute $\hat{F}(u, v)$ using Eq. (5.6-18). Obtain $\hat{\mathbf{f}}$ by taking the inverse Fourier transform of $\hat{F}(u, v)$.

Step 3. Form the residual vector $\mathbf{r}$ according to Eq. (5.6-19) and compute $\phi(\gamma) = \|\mathbf{r}\|^2$.

Step 4. Increment or decrement γ.

(a) $\phi(\gamma) < \|\mathbf{n}\|^2 - a$. Increment γ according to the algorithm given above or other appropriate method (such as a Newton–Raphson procedure).

(b) $\phi(\gamma) > \|\mathbf{n}\|^2 + a$. Decrement γ according to an appropriate algorithm.

Step 5. Return to step 2 and continue unless step 6 is true.

Step 6. $\phi(\gamma) = \|\mathbf{n}\|^2 \pm a$, where a determines the accuracy with which the constraint is satisfied. Stop the estimation procedure, with $\hat{\mathbf{f}}$ for the present value of γ being the restored image.

Constrained Lest Squares Reoration

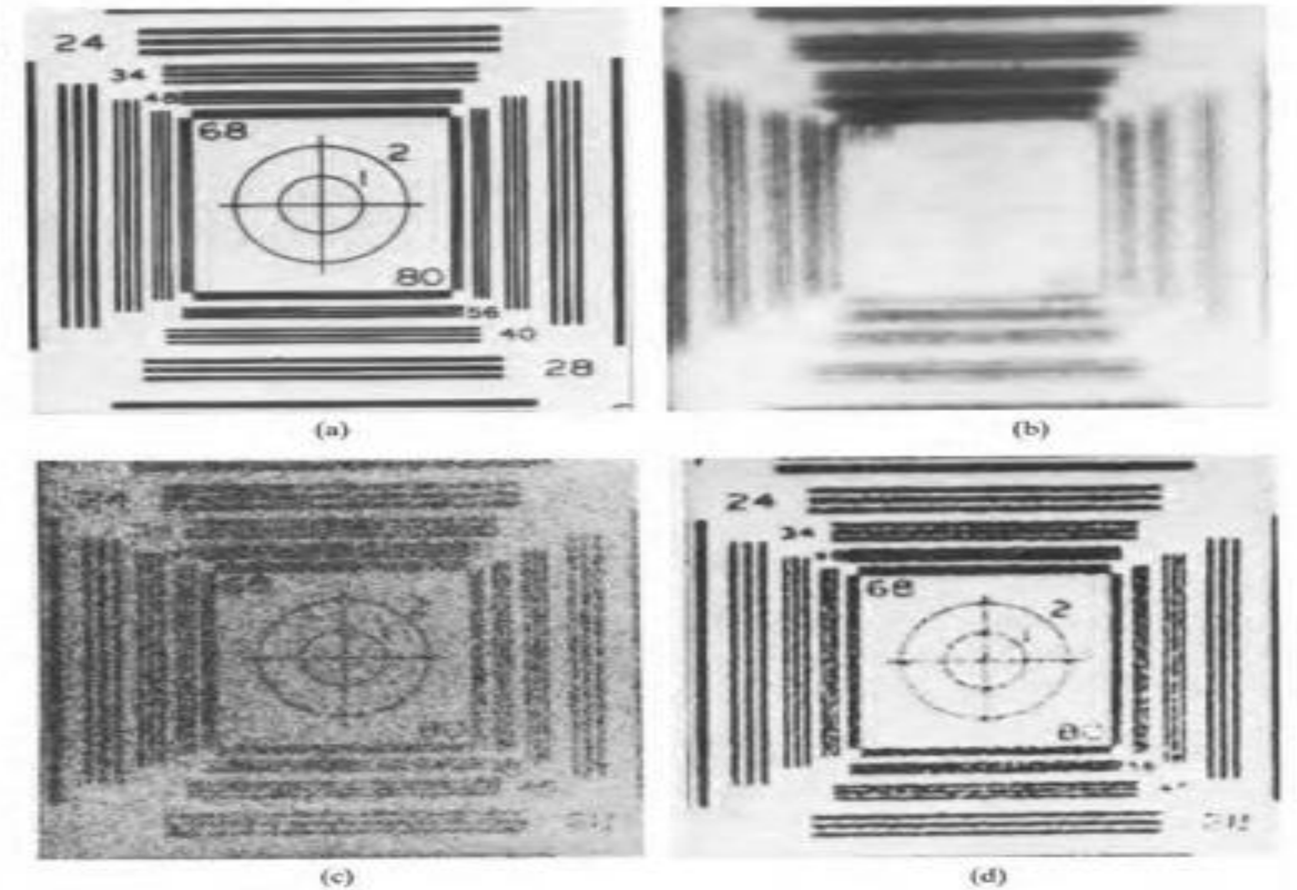

Figure 5.6 *(a) Original image; (b) image blurred and corrupted by additive noise; (c) image restored by inverse filtering; (d) image restored by the method of constrained least squares. (From Hunt [1973].)*

Interactive Restoration:

$$\eta(x, y) = A \sin(u_0 x + v_0 y).$$

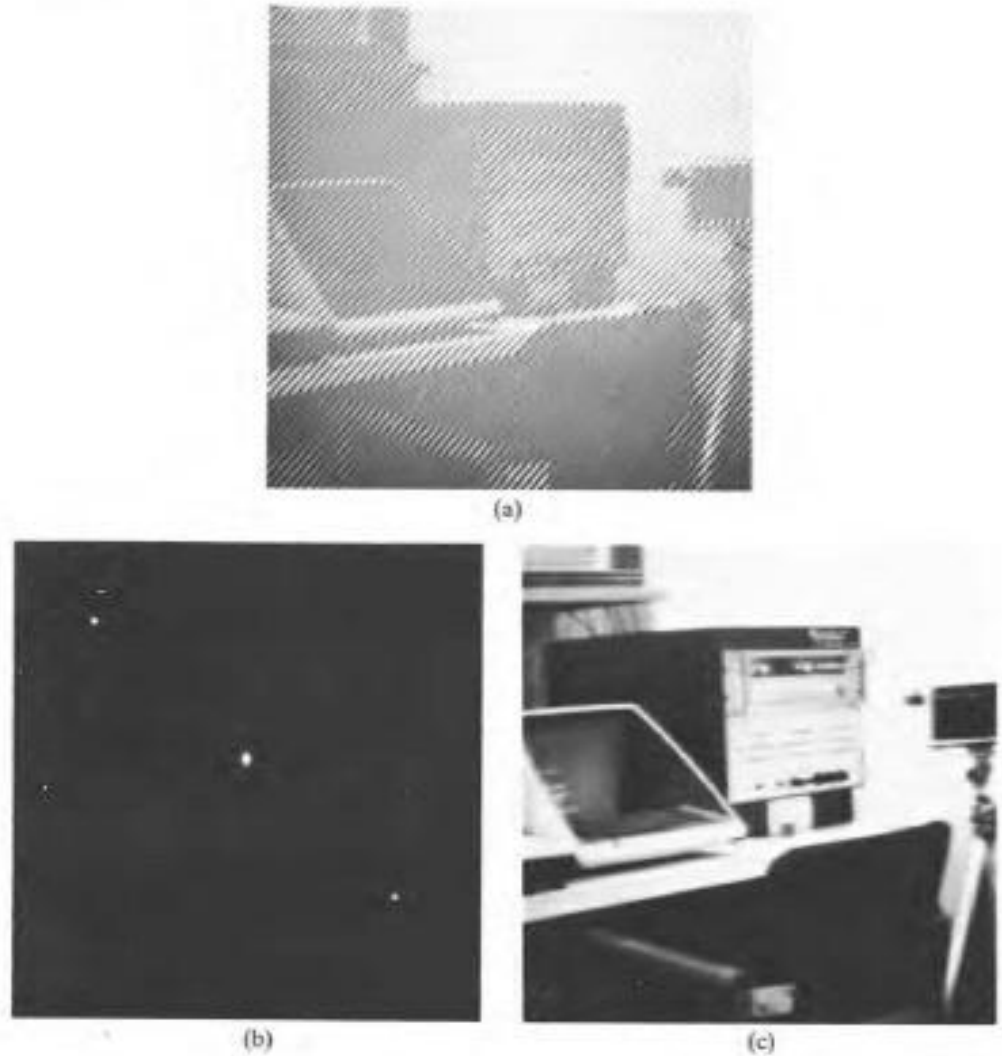

Figure 5.7 Example of sinusoidal interference removal: (a) corrupted image; (b) Fourier spectrum showing impulses due to sinusoidal pattern; (c) image restored by using a band-reject filter with a radius of 1.

Interactive Restoration:

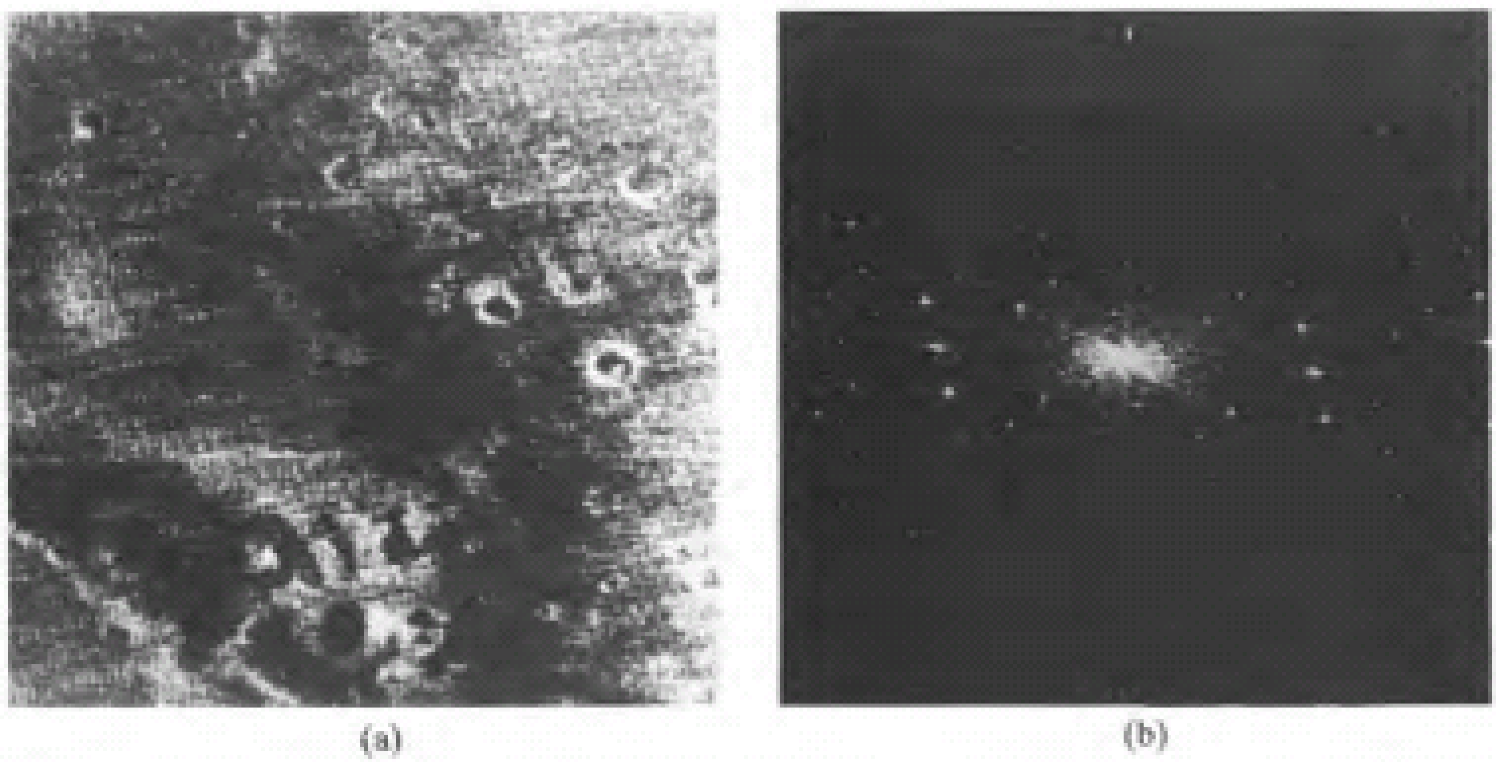

Figure 5.8 (a) Picture of the Martian terrain taken by Mariner 6; (b) Fourier spectrum. Note the periodic interference in the image and the corresponding spikes in the spectrum. (Courtesy of NASA, Jet Propulsion Laboratory.)

Interactive Restoration:

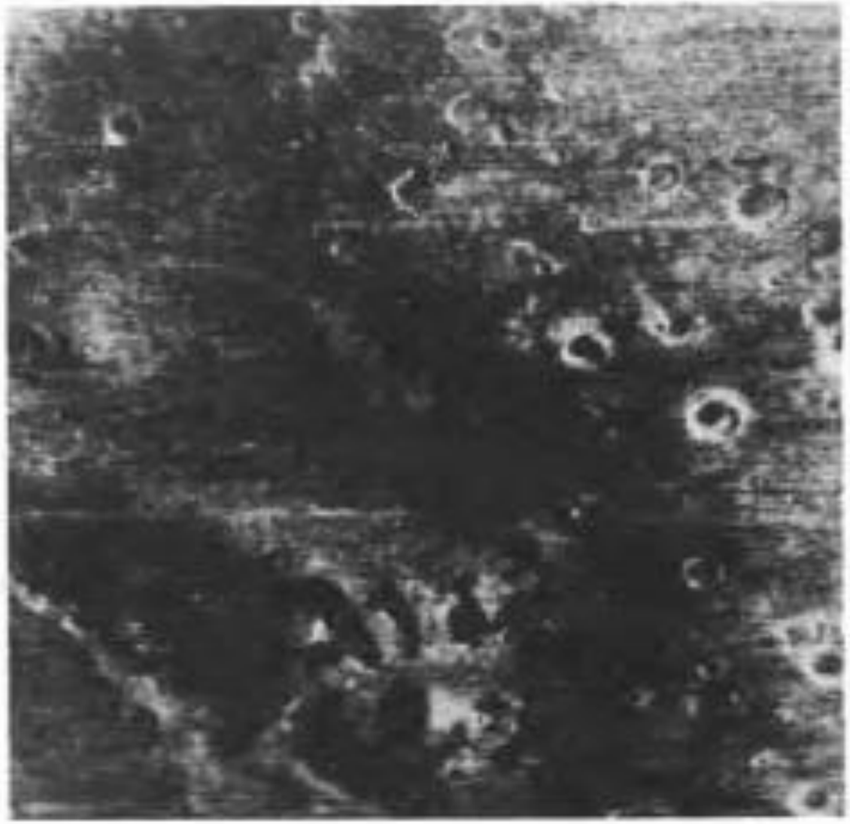

Figure 5.11 *Processed image. (Courtesy of NASA, Jet Propulsion Laboratory.)*

Restoration in the Spatial Domain

Geometric Transformations:

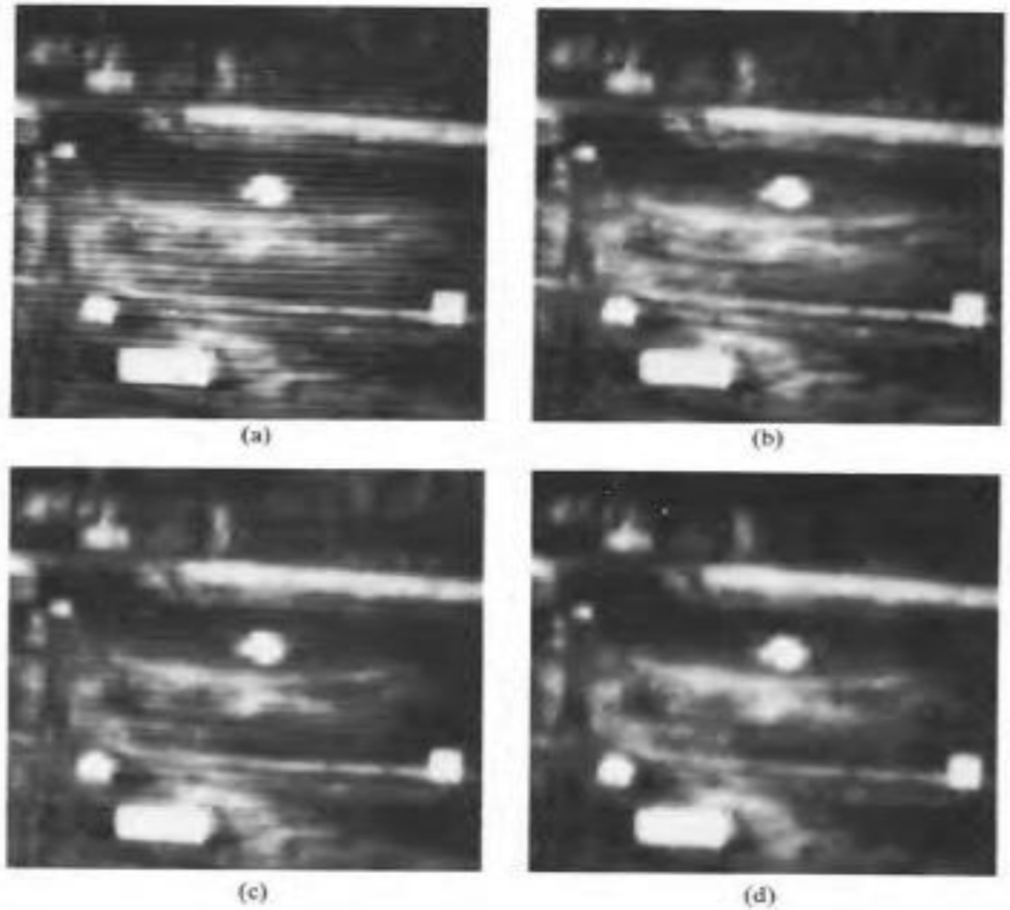

Figure 5.12 (a) Infrared image showing interference; (b) image restored using a notch filter in the frequency domain; (c) image restored using a 9 × 9 convolution mask; (d) result of applying the mask a second time. (From Meyer and Gonzalez [1983].)

Geometric Transformations:

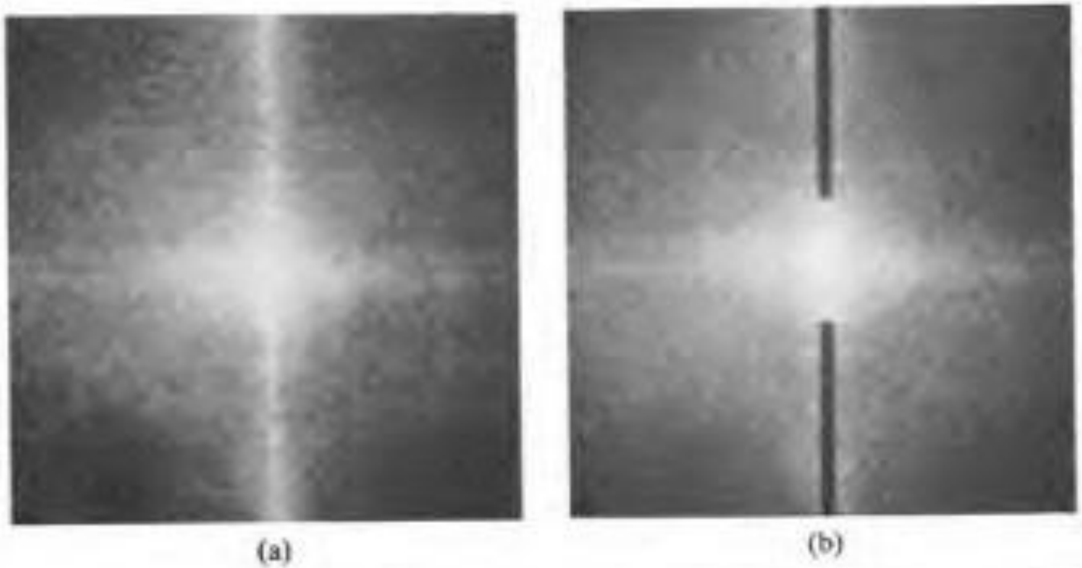

Figure 5.13 (a) Fourier spectrum of the image in Fig. 5.12(a); (b) Notch filter superimposed on the spectrum. (From Meyer and Gonzalez [1983].)

Spatial Transformations:

$$\hat{x} = r(x, y)$$

$$\hat{y} = s(x, y)$$

$$r(x, y) = c_1 x + c_2 y + c_3 xy + c_4 \quad (5.9\text{-}3)$$

and

$$s(x, y) = c_5 x + c_6 y + c_7 xy + c_8. \quad (5.9\text{-}4)$$

Then, from Eqs. (5.9-1) and (5.9-2),

$$\hat{x} = c_1 x + c_2 y + c_3 xy + c_4 \quad (5.9\text{-}5)$$

and

$$\hat{y} = c_5 x + c_6 y + c_7 xy + c_8. \quad (5.9\text{-}6)$$

Spatial Transformations:

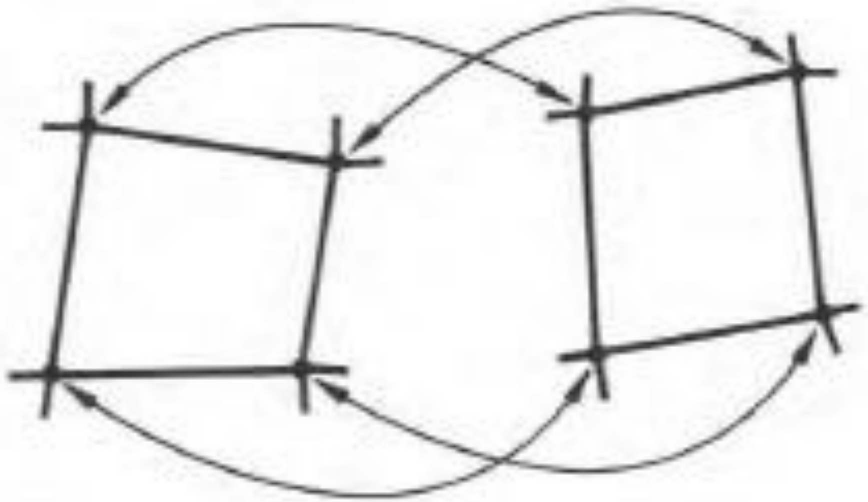

Figure 5.14 *Corresponding tiepoints in two image segments.*

Gray Level Interpolation:

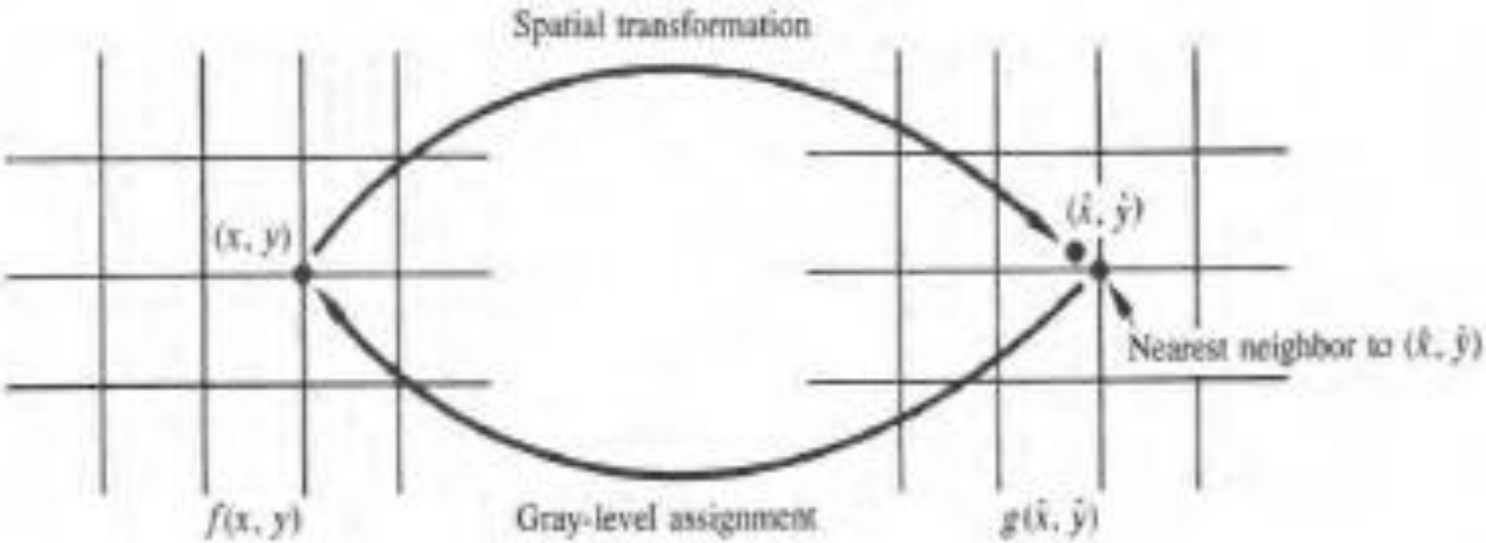

Figure 5.15 *Gray-level interpolation based on the nearest neighbor concept.*

Gray Level Interpolation:

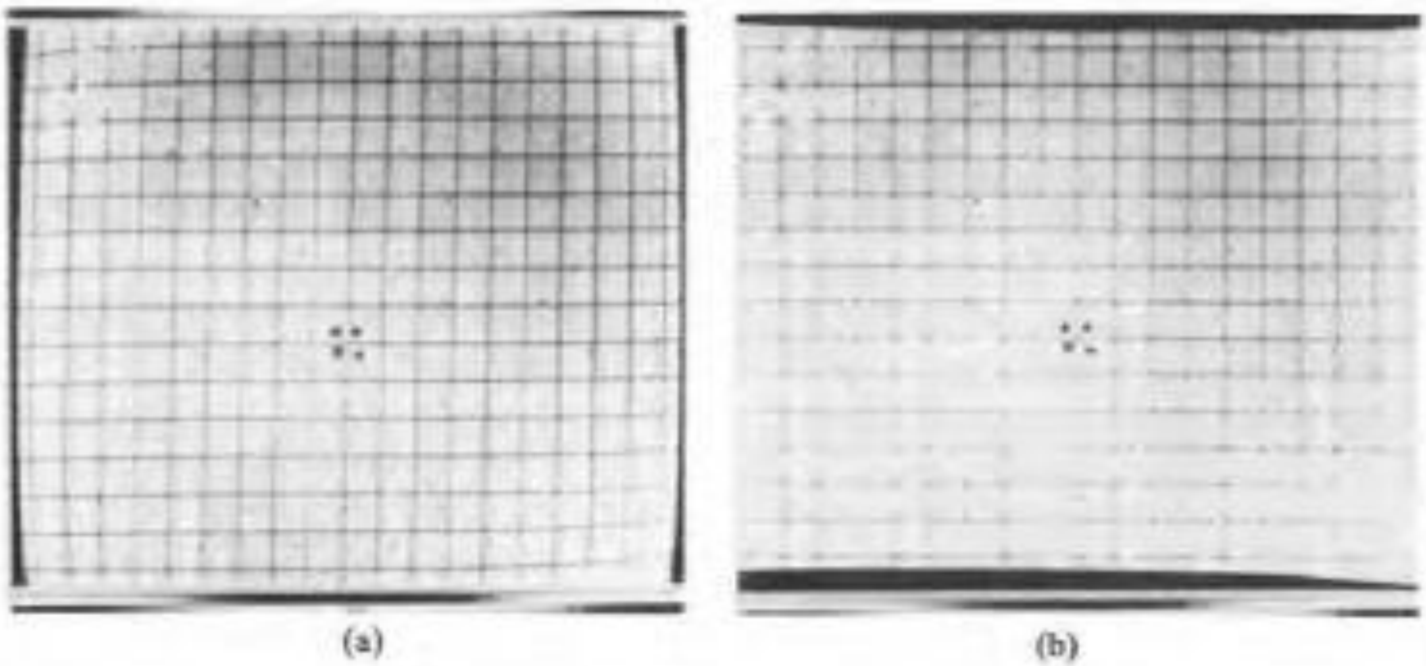

Figure 5.16 *(a) Distorted image; (b) image after geometric correction. (From O'Handley and Green [1972].)*