Digital Image Processing:

CONTINUOUS IMAGE MATHEMATICAL CHARACTERIZATION

- There are two basic mathematical characterizations of interest: deterministic and statistical.
- In *deterministic image representation*, a mathematical image function is defined and point properties of the image are considered.
- For a *statistical image representation, the* image is specified by average properties.

IMAGE REPRESENTATION

- Let $C(x, y, t, \lambda)$ represent the spatial energy distribution of an image source of radiant energy at spatial coordinates (x, y), at time t and wavelength λ.
- The physical imaging system also imposes some restriction on the maximum intensity of an image, for example, film saturation and *cathode ray tube* (*CRT*) phosphor heating. Hence it is assumed that

$$0 < C(x, y, t, \lambda) \leq A$$

CONTINUOUS IMAGE MATHEMATICAL CHARACTERIZATION

- For mathematical simplicity, all images are assumed to be nonzero only over a rectangular region for which

$$-L_x \leq x \leq L_x$$
$$-L_y \leq y \leq L_y$$

- The physical image is, of course, observable only over some finite time interval. Thus, let

$$-T \leq t \leq T$$

- The intensity response of a standard human observer to an image light function is commonly measured in terms of the instantaneous luminance of the light field as defined by

$$Y(x, y, t) = \int_0^\infty C(x, y, t, \lambda) V(\lambda) \, d\lambda$$

CONTINUOUS IMAGE MATHEMATICAL CHARACTERIZATION Cont…

where $V(\lambda)$ represents the *relative luminous efficiency function.*

For an arbitrary red–green–blue coordinate system, the instantaneous tristimulus values are

$$R(x, y, t) = \int_0^\infty C(x, y, t, \lambda) R_S(\lambda)\, d\lambda$$

$$G(x, y, t) = \int_0^\infty C(x, y, t, \lambda) G_S(\lambda)\, d\lambda$$

$$B(x, y, t) = \int_0^\infty C(x, y, t, \lambda) B_S(\lambda)\, d\lambda$$

where $R_S(\lambda)$, $G_S(\lambda)$, $B_S(\lambda)$ are spectral tristimulus values for the set of red, green and blue primaries.

TWO-DIMENSIONAL SYSTEMS

- A *two-dimensional system*, in its most general form, is simply a mapping of some input set of two-dimensional functions

  $F1(x, y), F2(x, y),..., F_N(x, y)$ to a set of output two-dimensional functions $G1(x, y), G2(x, y),..., G_M(x, y)$, Where $(-\infty < x, y < \infty)$ denotes the independent, continuous spatial variables of the functions.

  This mapping may be represented by the operators $O_m\{\cdot\}$ for $m = 1, 2,..., M$, which relate the input to output set of functions by the set of equations

$$\begin{bmatrix} G_1(x, y) = O_1\{F_1(x, y), F_2(x, y), \ldots, F_N(x, y)\} \\ \vdots \\ G_m(x, y) = O_m\{F_1(x, y), F_2(x, y), \ldots, F_N(x, y)\} \\ \vdots \\ G_M(x, y) = O_M\{F_1(x, y), F_2(x, y), \ldots, F_N(x, y)\} \end{bmatrix}$$

In specific cases, the mapping may be many-to-few, few-to-many, or one-to-one.The *one-to-one mapping* is defined as $G(x, y) = O\{F(x, y)\}$

Singularity Operators

- Singularity operators are widely employed in the analysis of two-dimensional systems, especially systems that involve sampling of continuous functions. The two dimensional *Dirac delta function* is a singularity operator that possesses the following properties:

$$\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \delta(x, y)\,dx\,dy = 1 \qquad \text{for } \varepsilon > 0$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi, \eta)\,\delta(x-\xi, y-\eta)\,d\xi\,d\eta = F(x, y)$$

In first eq. ε is an infinitesimally small limit of integration; The second eq. is called the *sifting property* of the Dirac delta function.

Singularity Operators Cont…

- The two-dimensional delta function can be decomposed into the product of two one-dimensional delta functions defined along orthonormal coordinates. Thus

$$\delta(x, y) = \delta(x)\delta(y)$$

The delta function also can be defined as a limit on a family of functions.

Additive Linear Operators

- A two-dimensional system is said to be an *additive linear system* if the system obeys the law of additive superposition. In the special case of one-to-one mappings, the additive superposition property requires that

$$O\{a_1F_1(x, y) + a_2F_2(x, y)\} = a_1O\{F_1(x, y)\} + a_2O\{F_2(x, y)\}$$

where $a1$ and $a2$ are constants that are possibly complex numbers.

A system input function $F(x, y)$ can be represented as a sum of amplitude weighted Dirac delta functions by the sifting integral,

$$F(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi, \eta)\,\delta(x-\xi, y-\eta)\,d\xi\,d\eta$$

Where $F(\xi, \eta)$ is the weighting factor of the impulse located at coordinates (ξ, η) in the x–y plane

Additive Linear Operators

- If the output of a general linear one-to-one system is defined to be

$$G(x, y) = O\{F(x, y)\}$$

then

$$G(x,y) = O\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi,\eta)\delta(x-\xi, y-\eta)\,d\xi\,d\eta\right\} \qquad (1.2\text{-}8a)$$

or

$$G(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi,\eta)O\{\delta(x-\xi, y-\eta)\}\,d\xi\,d\eta \qquad (1.2\text{-}8b)$$

Additive Linear Operators cont..

- In moving from Eq. 1.2-8*a* to Eq. 1.2-8*b*, the application order of the general linear operator and the integral operator have been reversed. Also, the linear operator has been applied only to the term in the integrand that is dependent on the spatial variables (*x, y*). The second term in the integrand of Eq. 1.2-8*b*, which is redefined as

$$H(x, y; \xi,\eta) \equiv O\{\delta(x - \xi, y - \eta)\}$$

is called the *impulse response* of the two-dimensional system. In optical systems, the impulse response is often called the *point spread function* of the system. Substitution of the impulse response function into Eq. 1.2-8*b* yields the additive *superposition integral*

Additive Linear Operators cont..

$$H(x, y; \xi, \eta) = H(x - \xi, y - \eta)$$

An additive linear two-dimensional system is called *space invariant* (isoplanatic) if its impulse response depends only on the factors $x - \xi$ and $y - \eta$.

For a space-invariant system

$$G(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi, \eta) H(x, y; \xi, \eta) \, d\xi \, d\eta$$

and the superposition integral reduces to the special case called the *convolution integral*, given by

$$G(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi, \eta) H(x - \xi, y - \eta) \, d\xi \, d\eta$$

Symbolically,

$$G(x, y) = F(x, y) \circledast H(x, y)$$

DIFFERENTIAL OPERATORS

- Edge detection in images is commonly accomplished by performing a spatial differentiation of the image field followed by a thresholding operation to determine points of steep amplitude change.
- Horizontal and vertical spatial derivatives are defined as

$$d_x = \frac{\partial F(x, y)}{\partial x}$$

$$d_y = \frac{\partial F(x, y)}{\partial y}$$

DIFFERENTIAL OPERATORS …cont

- Spatial second derivatives in the horizontal and vertical directions are defined as

$$d_{xx} = \frac{\partial^2 F(x,y)}{\partial x^2}$$

$$d_{yy} = \frac{\partial^2 F(x,y)}{\partial y^2}$$

- The sum of these two spatial derivatives is called the *Laplacian operator*

$$\nabla^2\{F(x,y)\} = \frac{\partial^2 F(x,y)}{\partial x^2} + \frac{\partial^2 F(x,y)}{\partial y^2}$$

TWO-DIMENSIONAL FOURIER TRANSFORM

- The two-dimensional *Fourier transform* of the image function $F(x, y)$ is defined as

$$\mathcal{F}(\omega_x, \omega_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x, y) \exp\{-i(\omega_x x + \omega_y y)\}\, dx\, dy$$

where ω_x and ω_y are *spatial frequencies and i*= $\sqrt{-1}$

TWO-DIMENSIONAL FOURIER TRANSFORM cont…

- In general, the Fourier coefficient $\mathcal{F}(\omega_x, \omega_y)$ is a complex number that may be represented in real and imaginary form,

$$\mathcal{F}(\omega_x, \omega_y) = \mathcal{R}(\omega_x, \omega_y) + i\mathcal{I}(\omega_x, \omega_y)$$

- or in magnitude and phase-angle form,

$$\mathcal{F}(\omega_x, \omega_y) = \mathcal{M}(\omega_x, \omega_y)\exp\{i\phi(\omega_x, \omega_y)\}$$

- where

$$\mathcal{M}(\omega_x, \omega_y) = [\mathcal{R}^2(\omega_x, \omega_y) + \mathcal{I}^2(\omega_x, \omega_y)]^{1/2}$$

$$\phi(\omega_x, \omega_y) = \arctan\left\{\frac{\mathcal{I}(\omega_x, \omega_y)}{\mathcal{R}(\omega_x, \omega_y)}\right\}$$

TWO-DIMENSIONAL FOURIER TRANSFORM cont…

- The input function *F(x, y) can be recovered from its Fourier transform by the inversion* formula

$$F(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}(\omega_x, \omega_y)\exp\{i(\omega_x x + \omega_y y)\}\, d\omega_x\, d\omega_y$$

- or in operator form $F(x,y) = O_{\mathcal{F}}^{-1}\{\mathcal{F}(\omega_x, \omega_y)\}$
- The functions $F(x, y)$ *and* $\mathcal{F}(\omega_x, \omega_y)$ *are called Fourier transform pairs.*

TWO-DIMENSIONAL FOURIER TRANSFORM
SEPARABILITY

- If the image function is spatially separable such that $F(x, y) = f_x(x) f_y(y)$ then $\mathcal{F}_y(\omega_x, \omega_y) = f_x(\omega_x) f_y(\omega_y)$

where $f_x(\omega_x)$ and $f_y(\omega_y)$ are one-dimensional Fourier transforms of $f_x(x)$ and $f_y(y)$, respectively. Also, if $F(x, y)$ and $\mathcal{F}(\omega_x, \omega_y)$ are two-dimensional Fourier transform pairs, the Fourier transform of $F^*(x, y)$ is $\mathcal{F}^*(-\omega_x, -\omega_y)$. An asterisk* used as a superscript denotes complex conjugation of a variable (i.e. if $F = A + iB$, then $F^* = A - iB$). Finally, if $F(x, y)$ is symmetric such that $F(x, y) = F(-x, -y)$, then $\mathcal{F}(\omega_x, \omega_y) = \mathcal{F}(-\omega_x, -\omega_y)$.

TWO-DIMENSIONAL FOURIER TRANSFORM
LINEARITY

- The Fourier transform is a linear operator. Thus

$$O_{\mathcal{F}}\{aF_1(x,y)+bF_2(x,y)\} = a\,\mathcal{F}_1(\omega_x,\omega_y)+b\,\mathcal{F}_2(\omega_x,\omega_y)$$

- where *a* and *b* are constants.

TWO-DIMENSIONAL FOURIER TRANSFORM
SCALING

- A linear scaling of the spatial variables results in an inverse scaling of the spatial frequencies as given by

$$O_{\mathcal{F}}\{F(ax, by)\} = \frac{1}{|ab|}\mathcal{F}\left(\frac{\omega_x}{a}, \frac{\omega_y}{b}\right)$$

- Hence, stretching of an axis in one domain results in a contraction of the corresponding axis in the other domain plus an amplitude change.

TWO-DIMENSIONAL FOURIER TRANSFORM
SHIFT

- A positional shift in the input plane results in a phase shift in the output plane:

$$O_{\mathcal{F}}\{F(x-a, y-b)\} = \mathcal{F}(\omega_x, \omega_y)\exp\{-i(\omega_x a + \omega_y b)\}$$

- Alternatively, a frequency shift in the Fourier plane results in the equivalence

$$O_{\mathcal{F}}^{-1}\{\mathcal{F}(\omega_x - a, \omega_y - b)\} = F(x, y)\exp\{i(ax + by)\}$$

TWO-DIMENSIONAL FOURIER TRANSFORM CONVOLUTION

- The two-dimensional Fourier transform of two convolved functions is equal to the products of the transforms of the functions. Thus; $O_{\mathcal{F}}\{F(x,y) \circledast H(x,y)\} = \mathcal{F}(\omega_x, \omega_y)\mathcal{H}(\omega_x, \omega_y)$
- The inverse theorem states that

$$O_{\mathcal{F}}\{F(x,y)H(x,y)\} = \frac{1}{4\pi^2}\mathcal{F}(\omega_x, \omega_y) \circledast \mathcal{H}(\omega_x, \omega_y)$$

TWO-DIMENSIONAL FOURIER TRANSFORM PARSEVAL'S THEOREM

- The energy in the spatial and Fourier transform domains is related by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(x,y)|^2\,dx\,dy = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\mathcal{F}(\omega_x,\omega_y)|^2\,d\omega_x\,d\omega_y$$

TWO-DIMENSIONAL FOURIER TRANSFORM
AUTOCORRELATION THEOREM

- The Fourier transform of the spatial autocorrelation of a function is equal to the magnitude squared of its Fourier transform. Hence

$$O_{\mathcal{F}}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\alpha,\beta)F^{*}(\alpha-x,\beta-y)\,d\alpha\,d\beta\right\} = \left|\mathcal{F}(\omega_x,\omega_y)\right|^2$$

TWO-DIMENSIONAL FOURIER TRANSFORM
SPATIAL DIFFERENTIALS

- The Fourier transform of the directional derivative of an image function is related to the Fourier transform by

$$O_{\mathcal{F}}\left\{\frac{\partial F(x,y)}{\partial x}\right\} = -i\omega_x \mathcal{F}(\omega_x, \omega_y)$$

$$O_{\mathcal{F}}\left\{\frac{\partial F(x,y)}{\partial y}\right\} = -i\omega_y \mathcal{F}(\omega_x, \omega_y)$$

- the Fourier transform of the Laplacian of an image function is equal to

$$O_{\mathcal{F}}\left\{\frac{\partial^2 F(x,y)}{\partial x^2} + \frac{\partial^2 F(x,y)}{\partial y^2}\right\} = -(\omega_x^2 + \omega_y^2)\,\mathcal{F}(\omega_x, \omega_y)$$

TWO-DIMENSIONAL FOURIER TRANSFORM *EXAMPLE*

- Consider an image function $F(x, y)$ that is the input to an additive linear system with an impulse response $H(x, y)$.
- The output image function is given by the convolution integral $G(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\alpha, \beta) H(x-\alpha, y-\beta)\, d\alpha\, d\beta$
- Taking the Fourier transform of both sides and reversing the order of integration on the right-hand side results in

$$G(\omega_x, \omega_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\alpha, \beta)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x-\alpha, y-\beta) \exp\{-i(\omega_x x + \omega_y y)\}\, dx\, dy\right] d\alpha\, d\beta$$

TWO-DIMENSIONAL FOURIER TRANSFORM *EXAMPLE* cont…

- The Fourier transform shift theorem, the inner integral is equal to the Fourier transform of multiplied by an exponential phase-shift $H(x, y)$ factor.
- Thus $\mathcal{G}(\omega_x, \omega_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\alpha, \beta)\mathcal{H}(\omega_x, \omega_y)\exp\{-i(\omega_x\alpha + \omega_y\beta)\}\, d\alpha\, d\beta$
- Performing the indicated Fourier transformation gives $\mathcal{G}(\omega_x, \omega_y) = \mathcal{H}(\omega_x, \omega_y)\mathcal{F}(\omega_x, \omega_y)$
- Then an inverse transformation provides the output image function

$$G(x, y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{H}(\omega_x, \omega_y)\mathcal{F}(\omega_x, \omega_y)\exp\{i(\omega_x x + \omega_y y)\}\, d\omega_x\, d\omega_y$$

IMAGE STOCHASTIC CHARACTERIZATION

- The statistical characterization of images assumes general familiarity with probability theory, random variables and stochastic processes. For continuous images, the image function $F(x, y, t)$ is assumed to be a member of a continuous three-dimensional stochastic process with space variables (x, y) and time variable (t).

The stochastic process $F(x, y, t)$ can be described completely by knowledge of its *joint probability density*

$$p\{F_1, F_2 ..., F_J; x_1, y_1, t_1, x_2, y_2, t_2, ..., x_J, y_J, t_J\}$$

for all sample points J, where (xj, yj, tj) represent space and time samples of image function $Fj(xj, yj, tj)$.