Implementation Example - FFT -

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Discrete Fourier Transform (DFT)

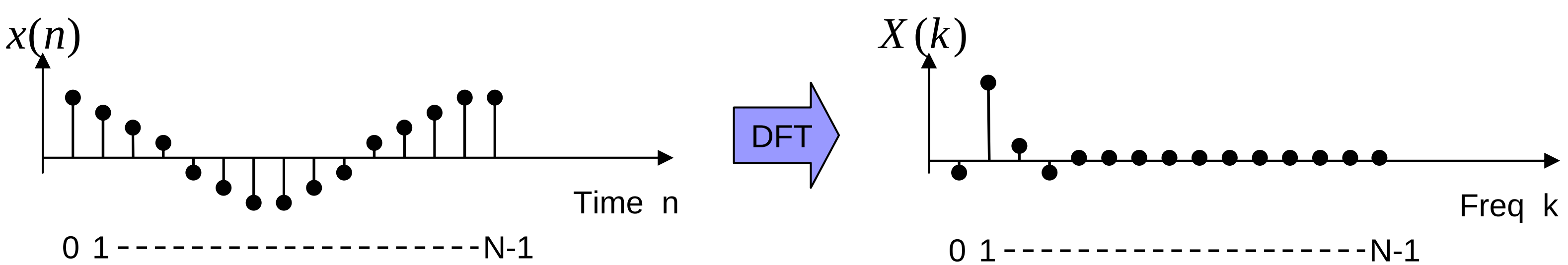
- DFT is transform from
  Time Domain Signal to Frequency Domain Signal.
- N point DFT is defined as follows

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\left(\frac{2\pi}{N}\right)nk} \quad (k = 0,1,\ldots,N-1)$$

$$W_N = e^{-j\left(\frac{2\pi}{N}\right)}$$

W_N is twiddle Factor!

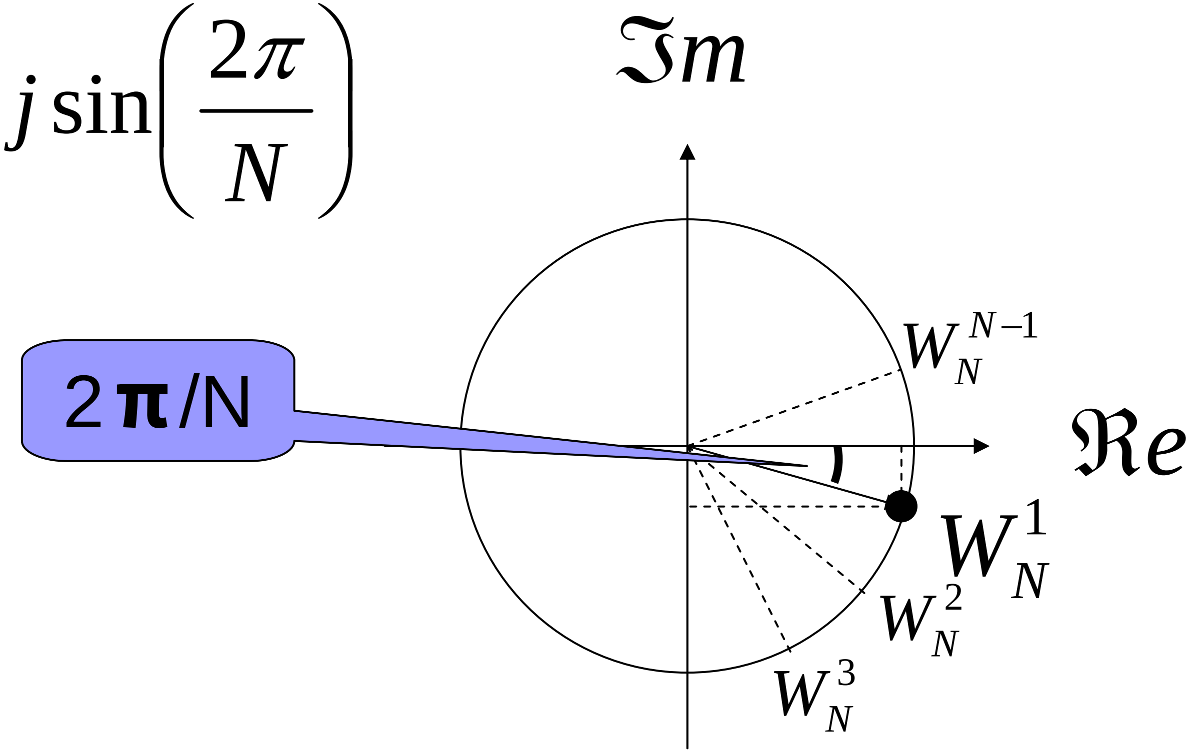
$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{nk} \quad (k = 0,1,\ldots,N-1)$$



What is Twiddle Factor W_N^{nk}

$$W_N = e^{-j\left(\frac{2\pi}{N}\right)}$$

$$= \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right)$$

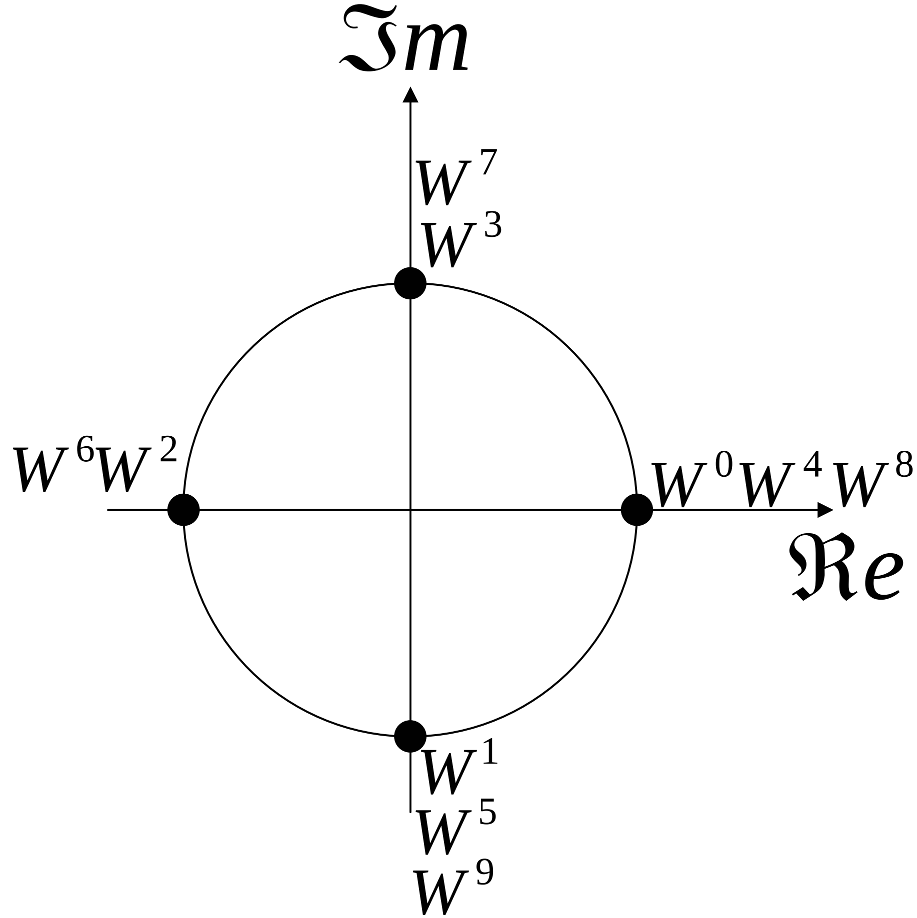


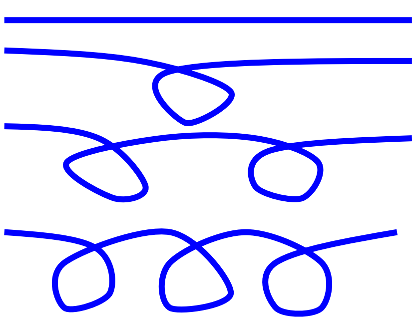
4-point DFT

$$W = e^{-j\left(\frac{2\pi}{4}\right)}$$

$$X(k) = \sum_{n=0}^{3} x(n)\cdot W^{nk} \quad (k = 0,1,2,3)$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 \\ W^0 & W^1 & W^2 & W^3 \\ W^0 & W^2 & W^4 & W^6 \\ W^0 & W^3 & W^6 & W^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$





- DFT operation is the group of inner products between complex sinusoids and x(n).

Simple DFT needs N^2 complex multiply and N(N-1) complex addition

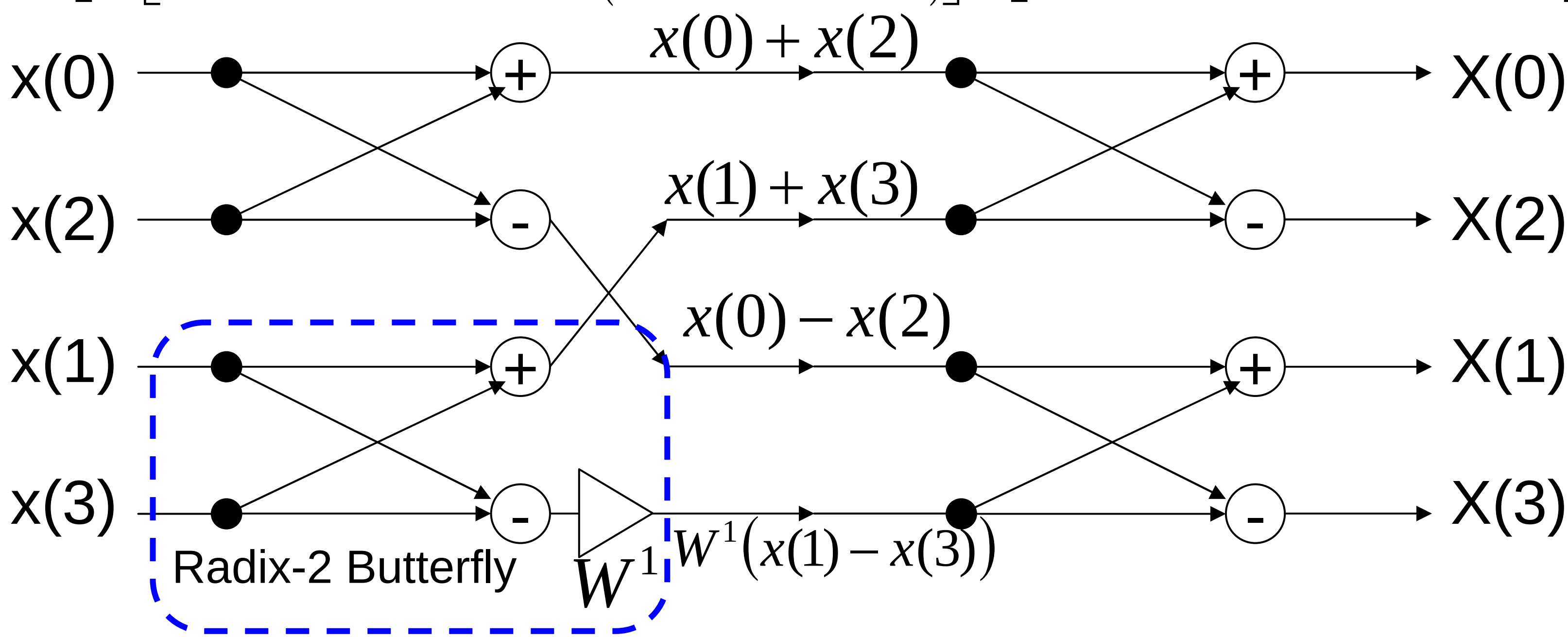
$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 \\ W^0 & W^1 & W^2 & W^3 \\ W^0 & W^2 & W^4 & W^6 \\ W^0 & W^3 & W^6 & W^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$= \begin{bmatrix} W^0 & W^0 & W^0 & W^0 \\ W^0 & W^2 & W^1 & W^3 \\ W^0 & W^4 & W^2 & W^6 \\ W^0 & W^6 & W^3 & W^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{bmatrix}$$

$$= \begin{bmatrix} W^0 & W^0 & W^0W^0 & W^0W^0 \\ W^0 & W^2 & W^1W^0 & W^1W^2 \\ W^0 & W^4 & W^2W^0 & W^2W^4 \\ W^0 & W^6 & W^3W^0 & W^3W^6 \end{bmatrix} \begin{bmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{bmatrix}$$

4 point FFT

$$
\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} =
\begin{bmatrix}
W^0x(0)+W^0x(2)+W^0\left(W^0x(1)+W^0x(3)\right) \\
W^0x(0)+W^2x(2)+W^1\left(W^0x(1)+W^2x(3)\right) \\
W^0x(0)+W^4x(2)+W^2\left(W^0x(1)+W^4x(3)\right) \\
W^0x(0)+W^6x(2)+W^3\left(W^0x(1)+W^6x(3)\right)
\end{bmatrix} =
\begin{bmatrix}
x(0)+x(2)+\left(x(1)+x(3)\right) \\
x(0)-x(2)+W^1\left(x(1)-x(3)\right) \\
x(0)+x(2)-\left(x(1)+x(3)\right) \\
x(0)-x(2)-W^1\left(x(1)-x(3)\right)
\end{bmatrix}
$$



This is Fast Fourier Transform Algorithm!
Only 4 butterfly operation for 4-FFT. 8 addition + 4 multiply.
DFT needs 12 addition + 16 multiply.

Radix-2 Butterfly

$$X(k) = \sum_{n=0}^{N} x(n) \cdot W_N^{nk}$$

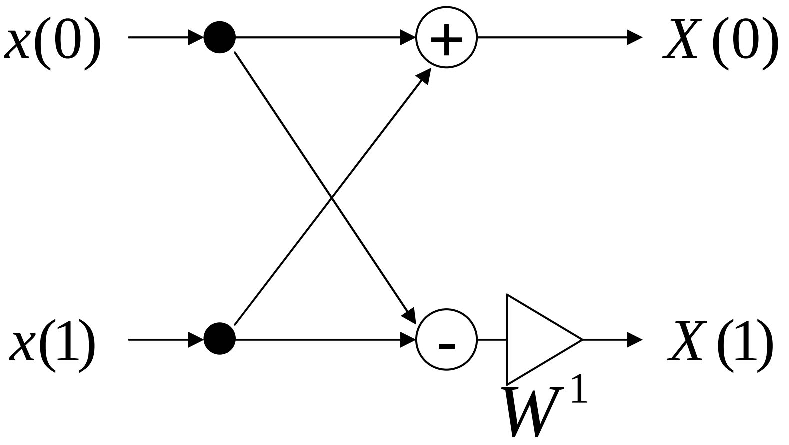
$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) \cdot W_N^{nk} + \sum_{n=\frac{N}{2}}^{N-1} x(n) \cdot W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) \cdot W_N^{nk} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) \cdot W_N^{\left(n+\frac{N}{2}\right)k}$$

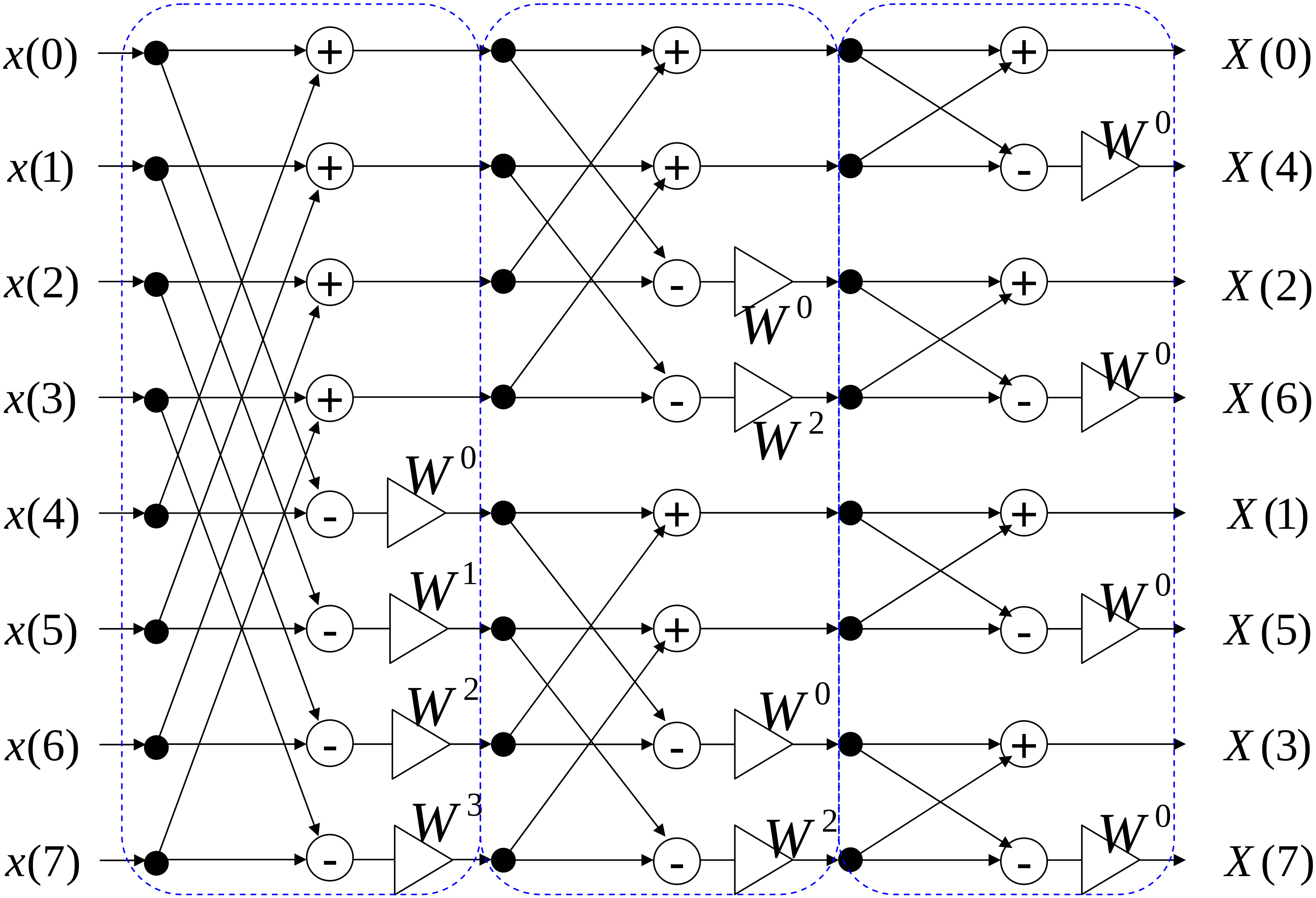
$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) \cdot W_N^{nk} + W_N^{\frac{N}{2}k} \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) \cdot W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) \cdot W_N^{nk} + (-1)^k \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) \cdot W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left\{ x(n) + (-1)^k x\left(n + \frac{N}{2}\right) \right\} W_N^{nk}$$



8-point FFT by radix-2 butterfly



Radix-4 Butterfly

$$X(k)=\sum_{n=0}^{N}x(n)\cdot W_N^{nk}$$

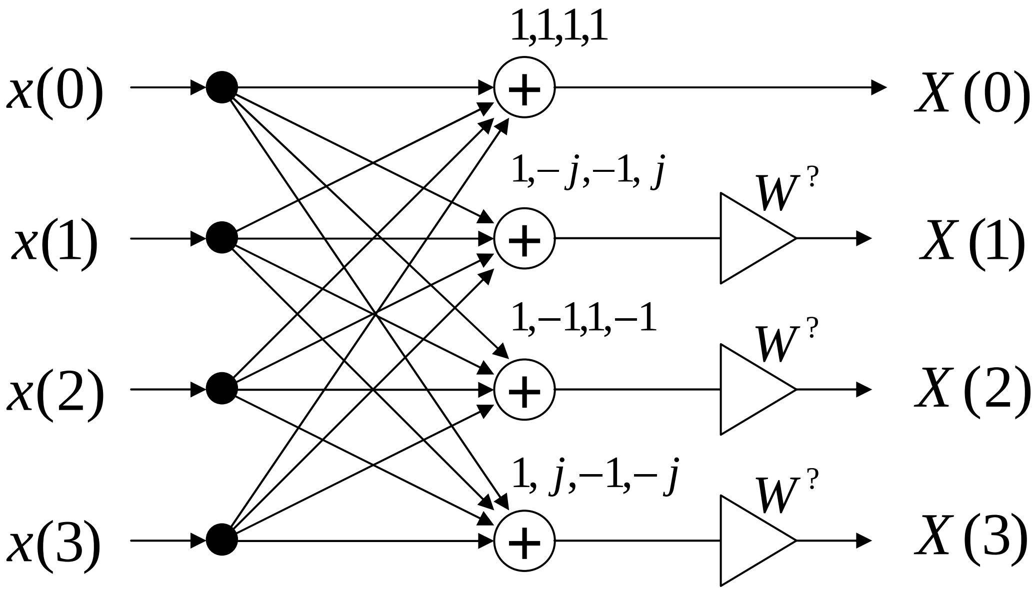
$$=\sum_{n=0}^{\frac{N}{4}-1}x(n)\cdot W_N^{nk}+\sum_{n=\frac{N}{4}}^{\frac{2N}{4}-1}x(n)\cdot W_N^{nk}+\sum_{n=\frac{2N}{4}}^{\frac{3N}{4}-1}x(n)\cdot W_N^{nk}+\sum_{n=\frac{3N}{4}}^{N-1}x(n)\cdot W_N^{nk}$$

$$=\sum_{n=0}^{\frac{N}{4}-1}x(n)\cdot W_N^{nk}+W_N^{\frac{N}{4}k}\sum_{n=0}^{\frac{N}{2}-1}x\left(n+\frac{N}{4}\right)\cdot W_N^{nk}+W_N^{\frac{2N}{4}k}\sum_{n=0}^{\frac{N}{2}-1}x\left(n+\frac{2N}{4}\right)\cdot W_N^{nk}+W_N^{\frac{3N}{4}k}\sum_{n=0}^{\frac{N}{2}-1}x\left(n+\frac{3N}{4}\right)\cdot W_N^{nk}$$

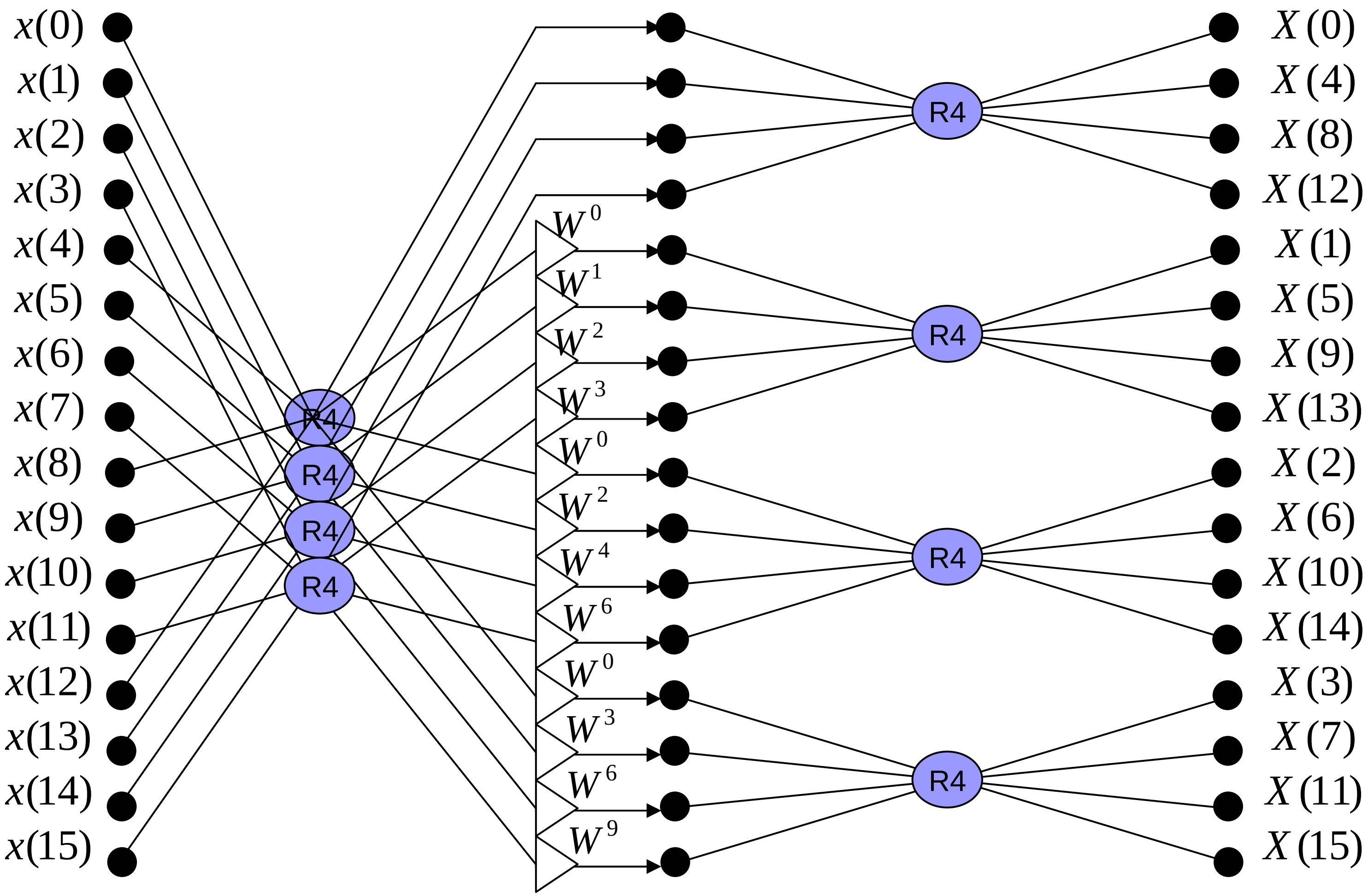
$$=\sum_{n=0}^{\frac{N}{4}-1}\left\{x(n)+(-j)^k x\left(n+\frac{N}{4}\right)+(-1)^k x\left(n+\frac{2N}{4}\right)+(j)^k x\left(n+\frac{3N}{4}\right)\right\}W_N^{nk}$$

Radix-4 Butterfly (2)

$$X(k)=\sum_{n=0}^{\frac{N}{4}-1}\left\{x(n)+(-j)^k x\left(n+\frac{N}{4}\right)+(-1)^k x\left(n+\frac{2N}{4}\right)+(j)^k x\left(n+\frac{3N}{4}\right)\right\}W_N^{nk}$$



16-FFT by radix-4 butterfly

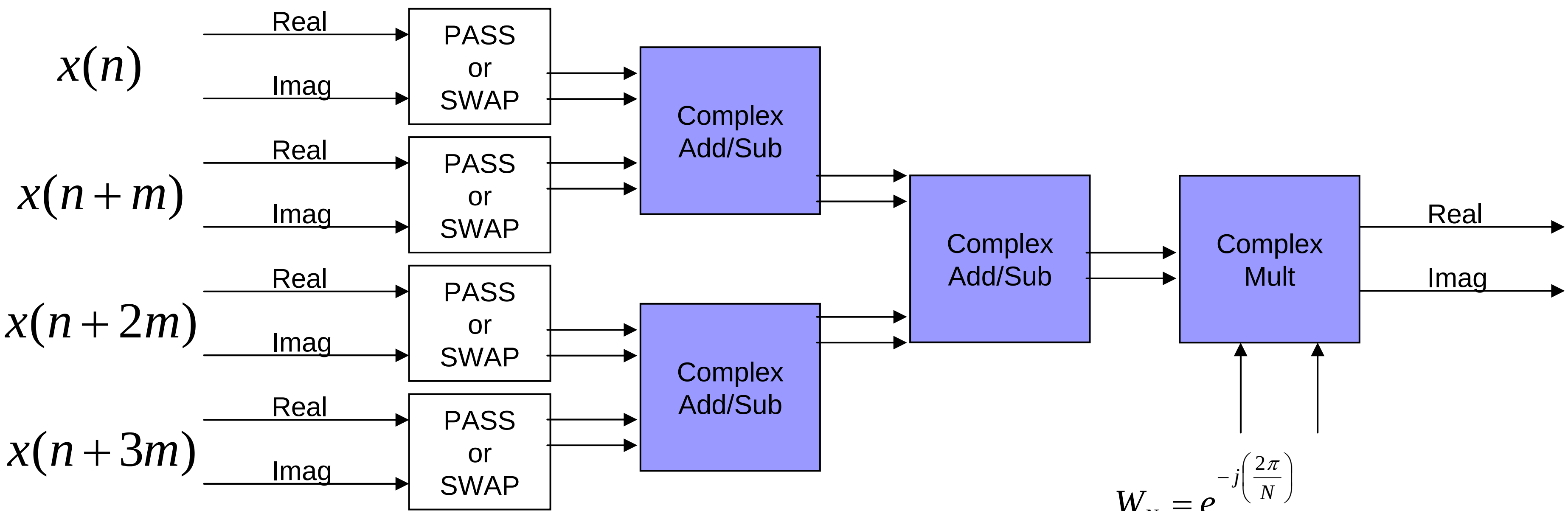


Architecture Comparison

- 1KFFT
  - Radix-2 implementation
    - 2^{10}=1024 : 10 stages
    - Each Stage needs 512 radix-2
    - Each radix-2 needs 2 complex addition and 1 complex multiply
    - Total : 10240 complex addition + 5120 complex multiply
  - Radix-4 omplementation
    - 4^5=1024 : 5 stages
    - Each Stage needs 256 radix-4
    - Each radix-4 needs 12 complex addition and 3 complex multiply
    - Total : 15360 complex addition + 3840 complex multiply

  - Usually multiply is expensive than addition
  - Then, radix-4 implementation is better!

Radix-4 computation unit

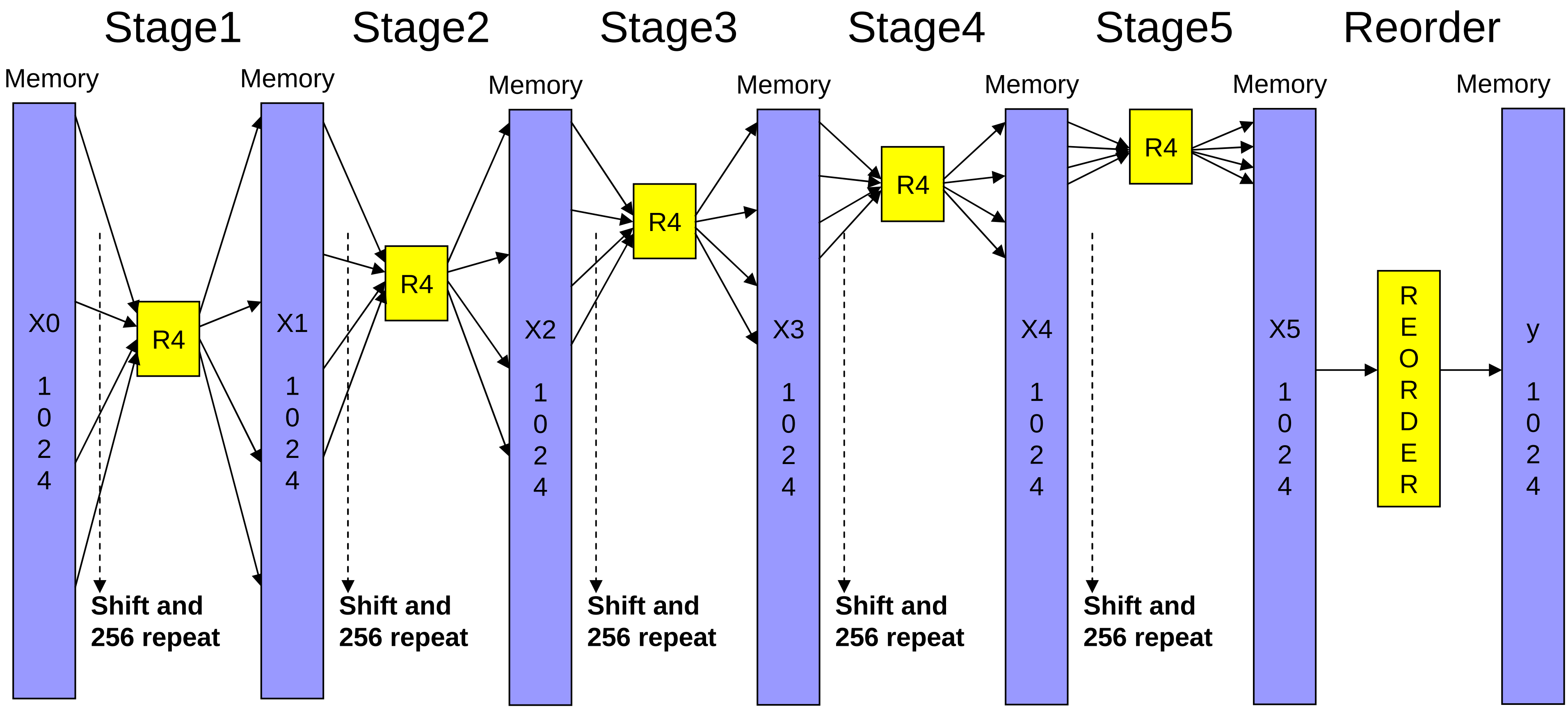


$$W_N = e^{-j\left(\frac{2\pi}{N}\right)}$$

$$= \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right)$$

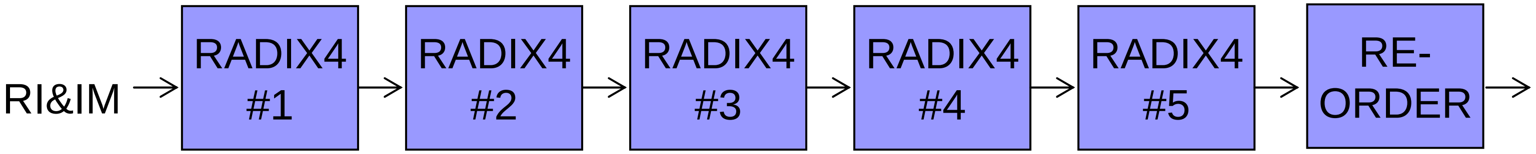
- complex add : (a+bj)+(c+dj)= (a+c)+(b+d)j
  - 1 complex add = 2 real add
- Complex mult : (a+bj)(c+dj) = (ac-bd)+(bc+ad)j
  - 1 complex mult = 4 real mult + 2 real add

1K FFT architecture

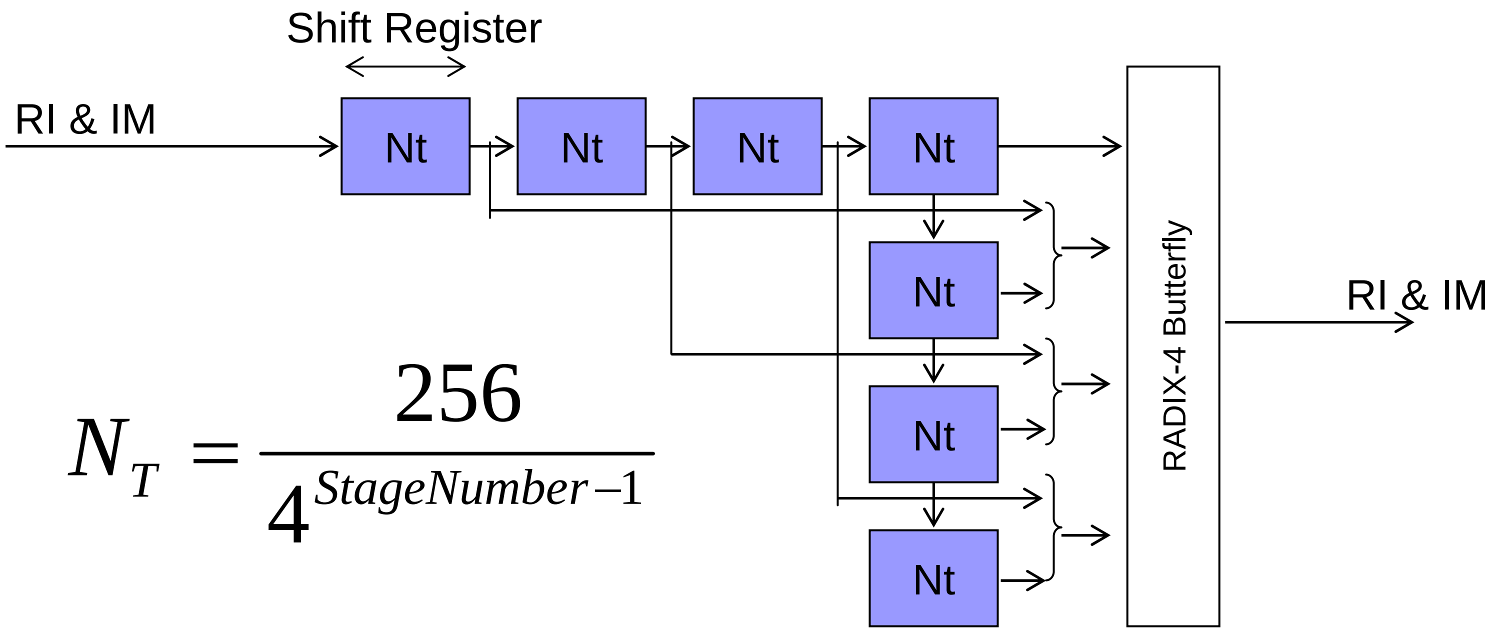


See code "myfft1024.m"

Serial Implementation



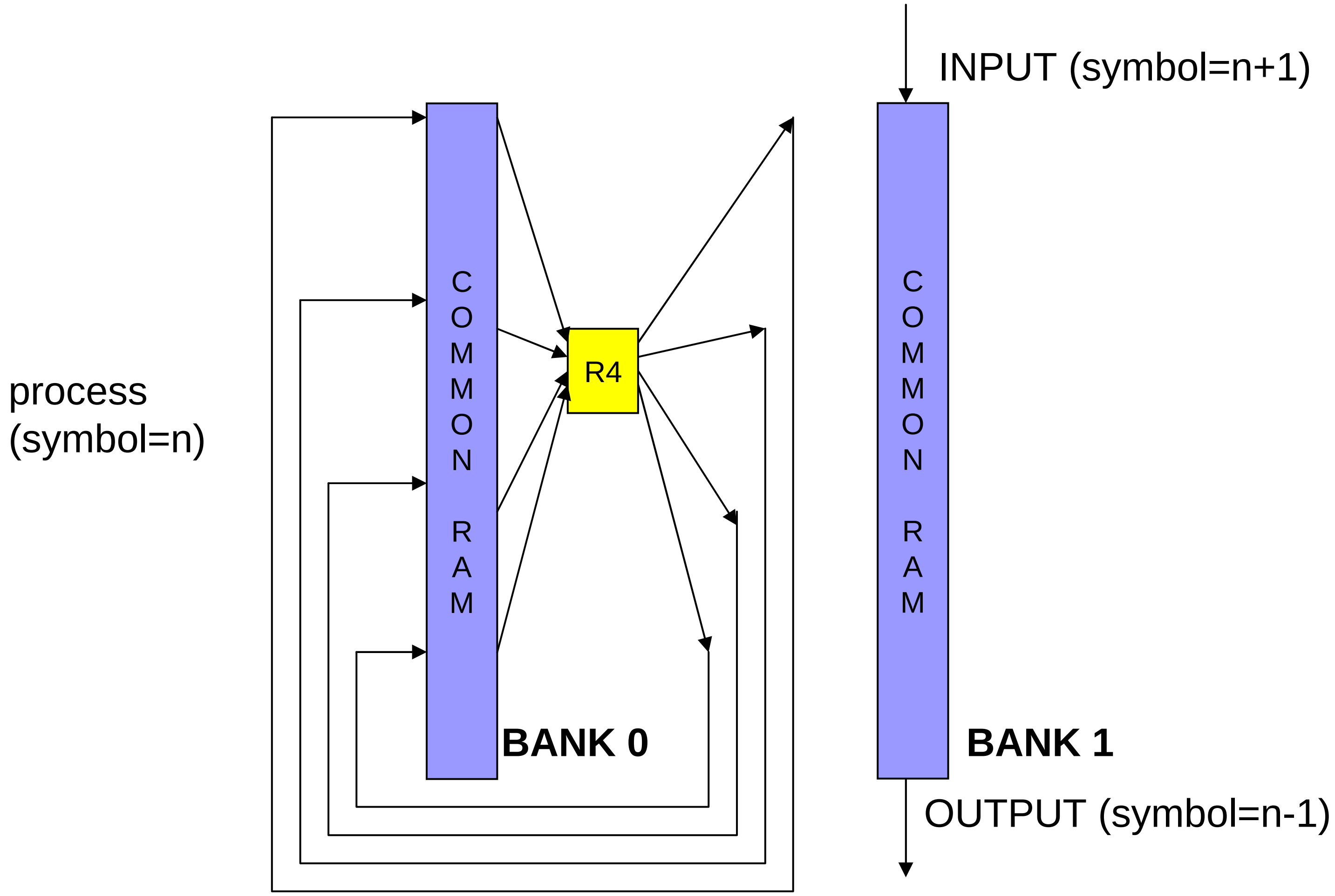
Each Stage



$$N_T = \frac{256}{4^{StageNumber-1}}$$

Loop Implementation

- Loop implementation is just to use common RAM for x0, x1, x2, …, y
- Use Two Bank RAM for parallel I/O and computation



Final Report Task

- Using library or internet, find out a digital system employing Digital Signal Processor or Programmable Devices for system construction. And Explain what is the application, why those devices are used in the system (if DSP is used, why they did not choose FPGA instead), and how the devices used in the system.
- Report should A4 pages with enough explanation and figures.
- Deadline is August 10th by email to wada@ie.u-ryukyu.ac.jp in PDF format.
- Mail title : SysArchFinalRep06(your name)