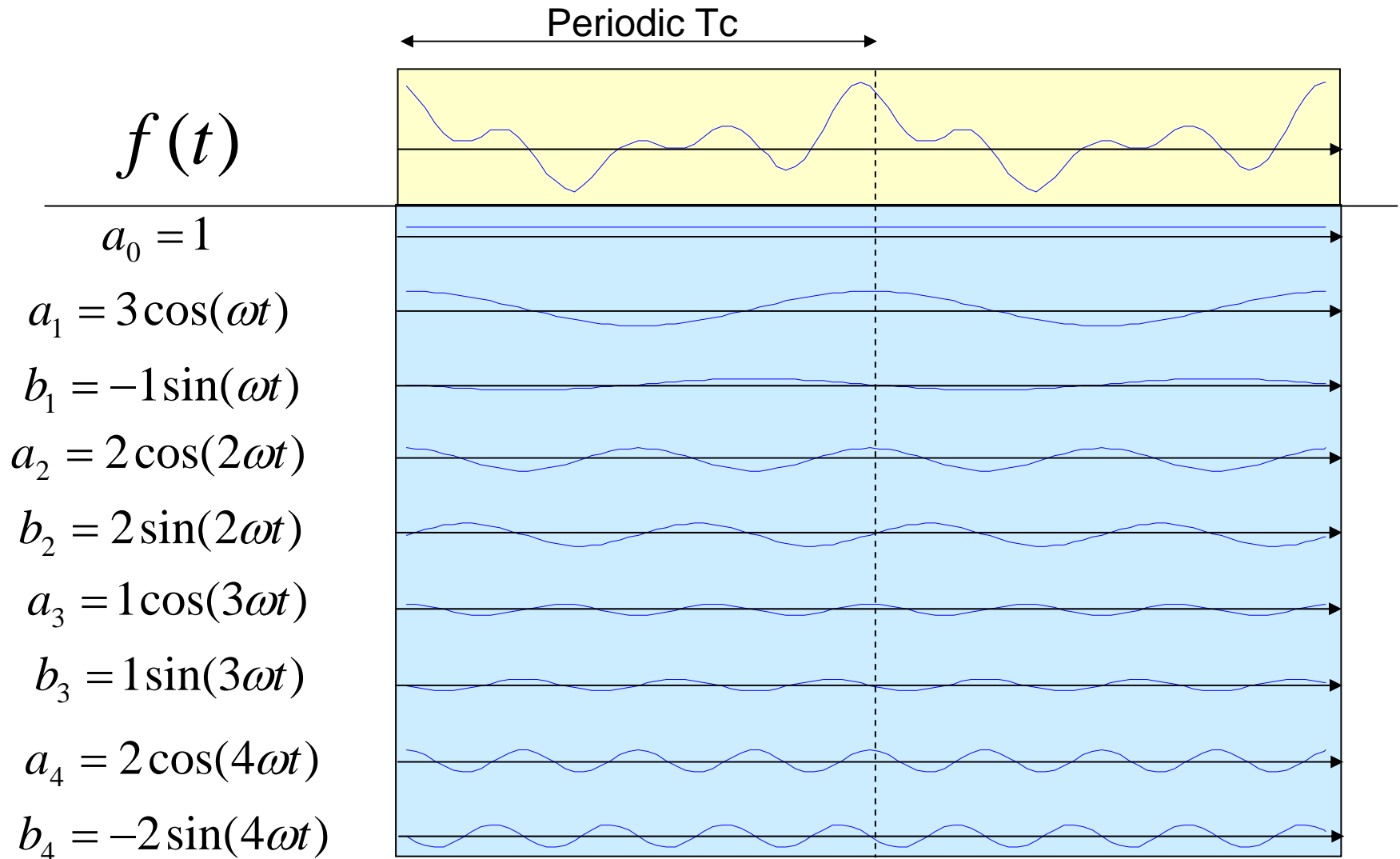

OUTLINE

- Periodic Signal
- Fourier series introduction
- Sinusoids Orthogonality
- Integration vs inner product

Consider any wave is sum of simple sin and cosine



Periodic signal is composed of DC + same frequency sinusoid + multiple frequency sinusoids

$$f(t) =$$

1

$$+ 3 \cos(\omega t) - 1 \sin(\omega t)$$

$$+ 2 \cos(2\omega t) + 2 \sin(2\omega t)$$

$$+ 1 \cos(3\omega t) + 1 \sin(3\omega t)$$

$$+ 2 \cos(4\omega t) - 2 \sin(4\omega t)$$

$$\omega = 2\pi f_c = 2\pi \frac{1}{T_c}$$

Frequency = 0 Hz

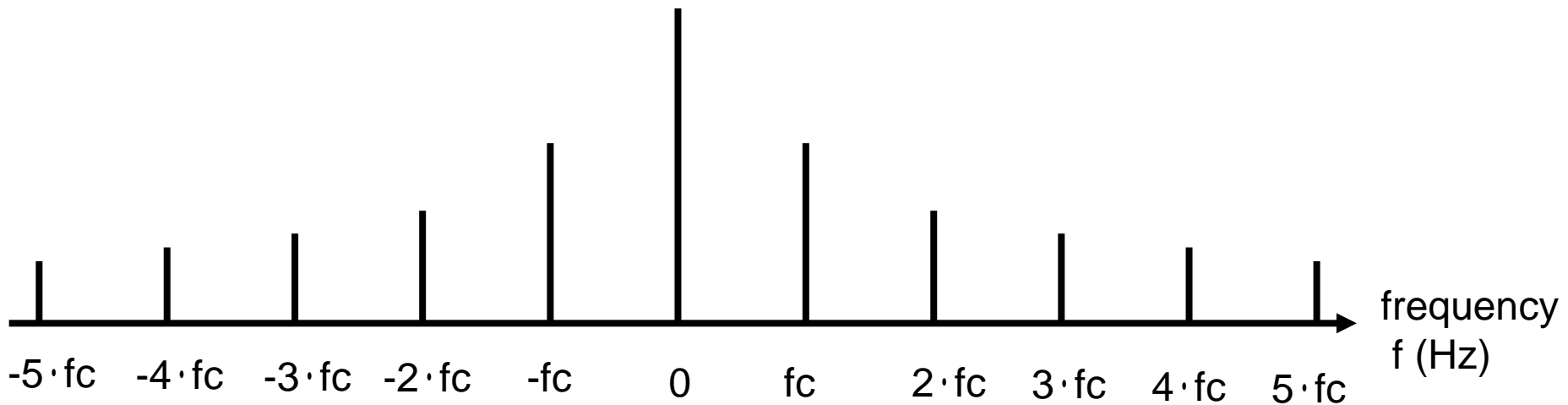
Basic frequency $f_c = 1/T_c$

2 x f_c

3 x f_c

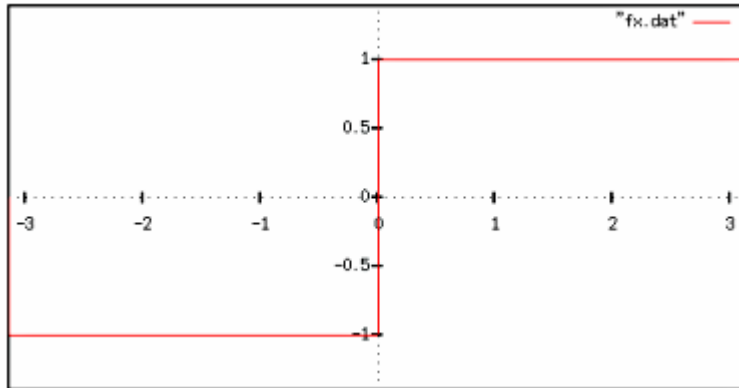
4 x f_c

Spectrum of periodic signal



There are only $n * fc$ (n =integer) frequencies!

Another example (even rectangular pulse)

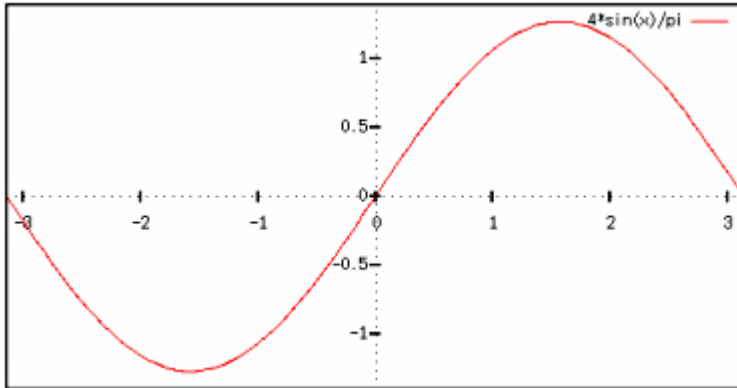


$$f(x) = \begin{cases} -1 & (-\pi < x < 0) \\ 1 & (0 < x < \pi) \end{cases}, \quad f(x + 2\pi) = f(x)$$

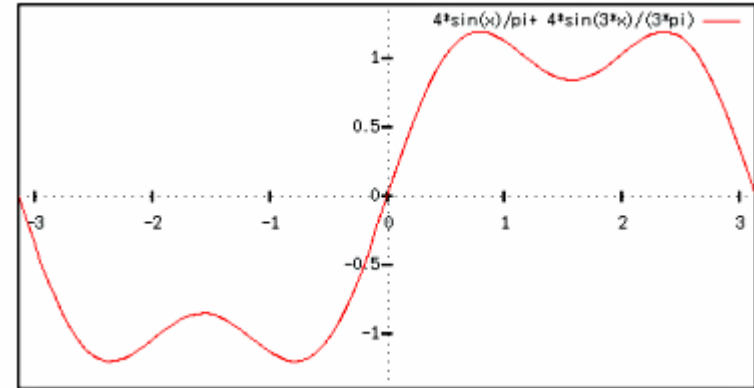
$$\begin{aligned} f(x) &= \frac{4}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \frac{1}{7} \sin 7x + \cdots \right) \\ &= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin\{(2n-1)x\}}{2n-1} \end{aligned}$$

Increase the number of sum (1)

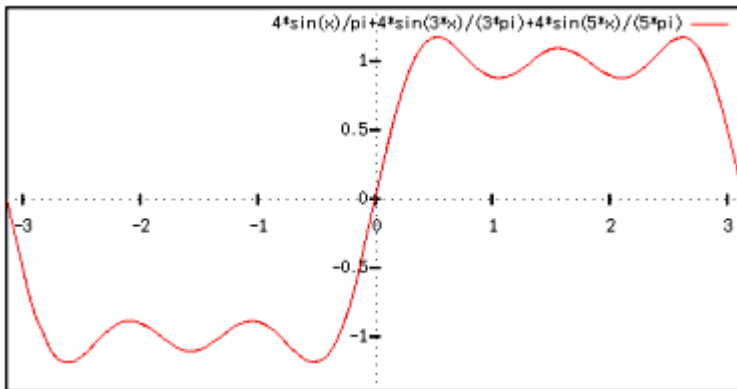
N=1



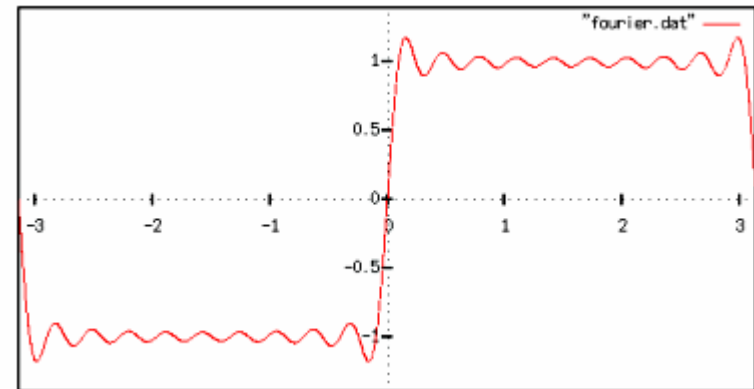
N=2



N=3

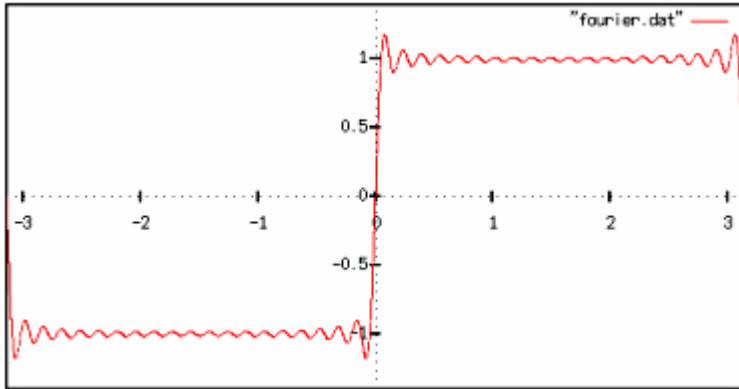


N=10

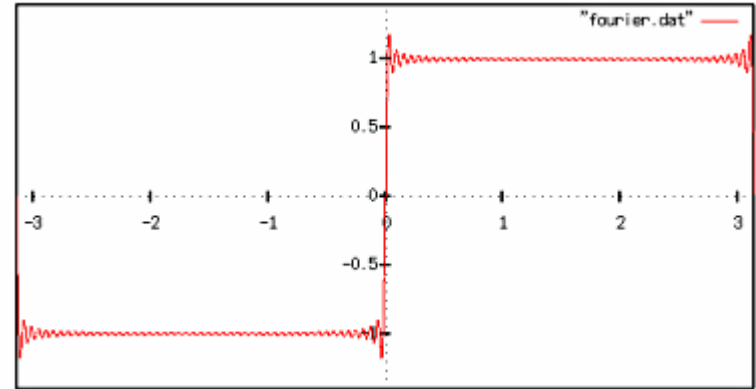


Increase the number of sum (2)

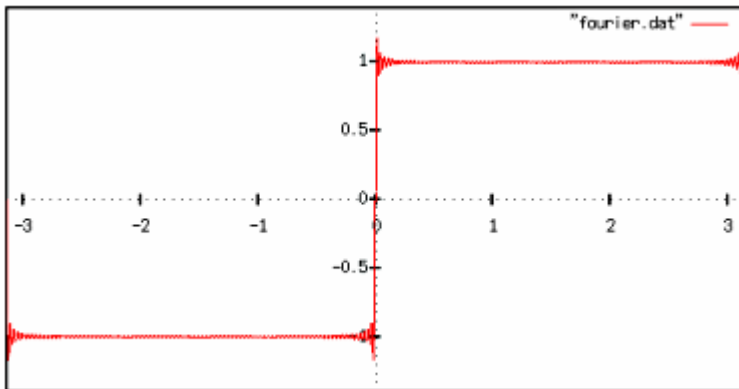
N=20



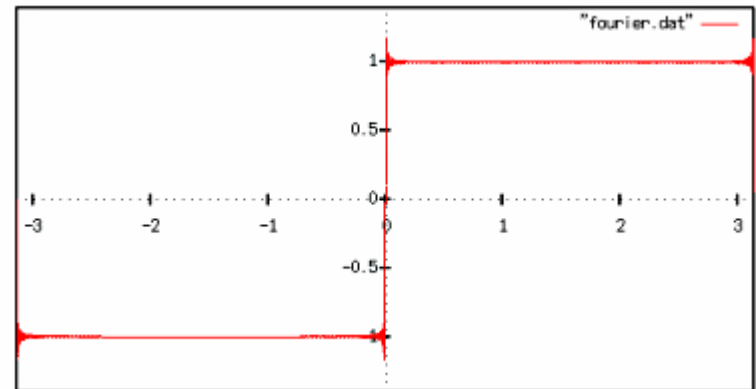
N=50



N=100



N=200



Fourier

- Jean Baptiste Joseph, Baron de Frourier
France, 1778/Mar/21 – 1830/May/16
- Fourier Series paper is written in 1807
- Even discontinue function (such as rectangular pulse) can be composed of many sinusoids.
- Nobody believed the paper at that time.



Fourier Series

- If $f(t)$'s period is T_c ... $\omega = 2\pi f_c = 2\pi \frac{1}{T_c}$

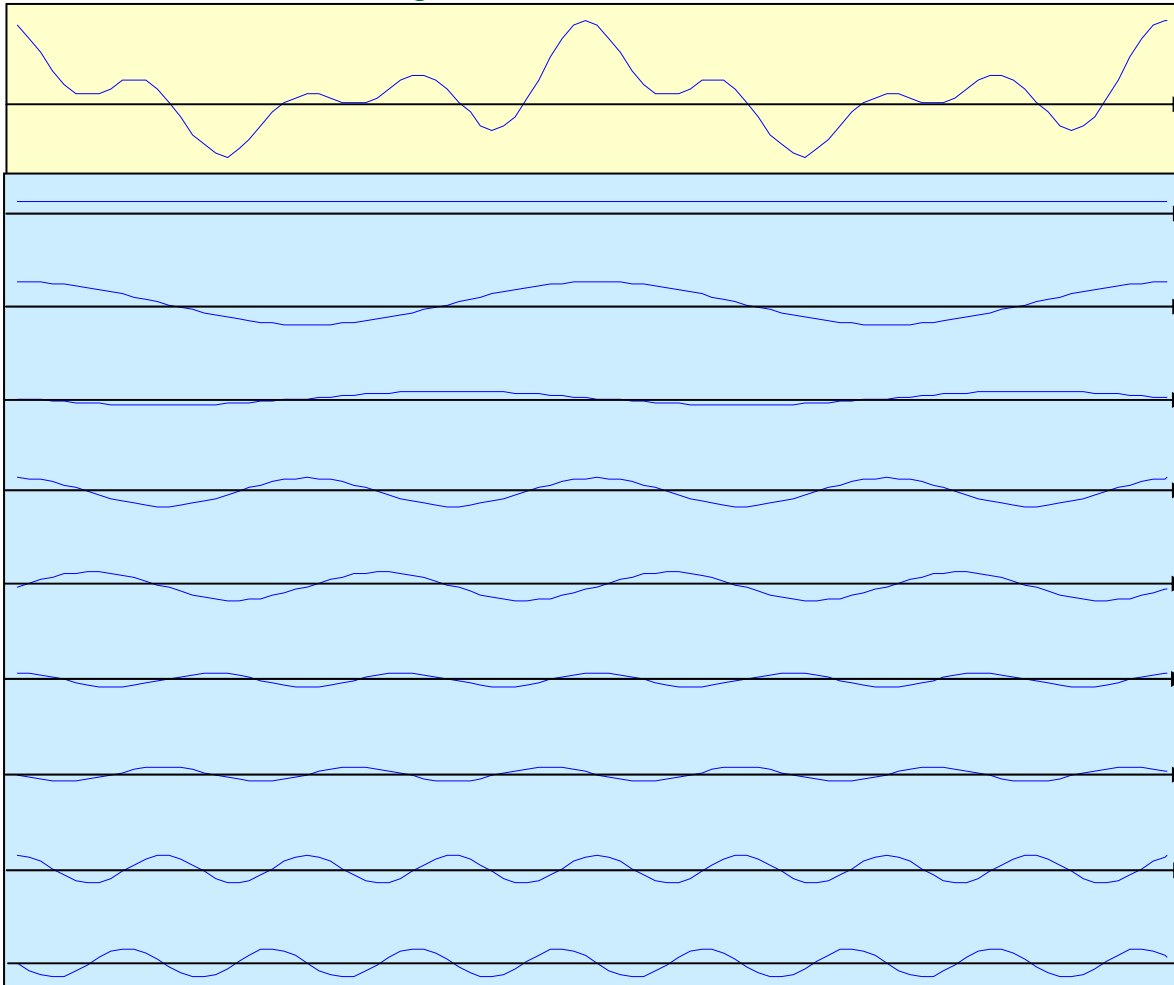
$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t))$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n \frac{2\pi}{T_c} t) + b_n \sin(n \frac{2\pi}{T_c} t))$$

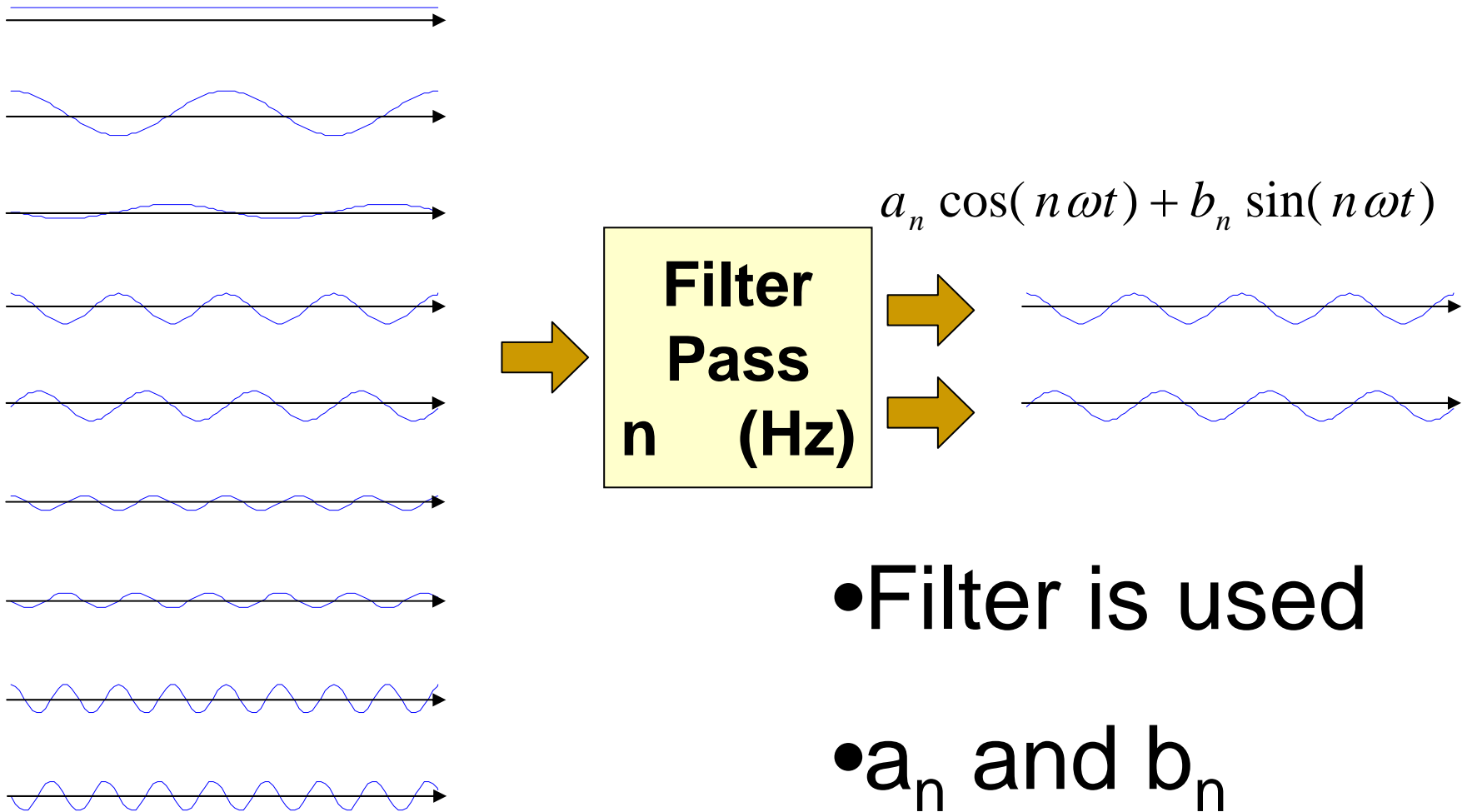
- If we use complex exponential....,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{jn\omega t}$$

Anyway, when you see the periodic signal,
Please think it is just sum of sinusoids!!!

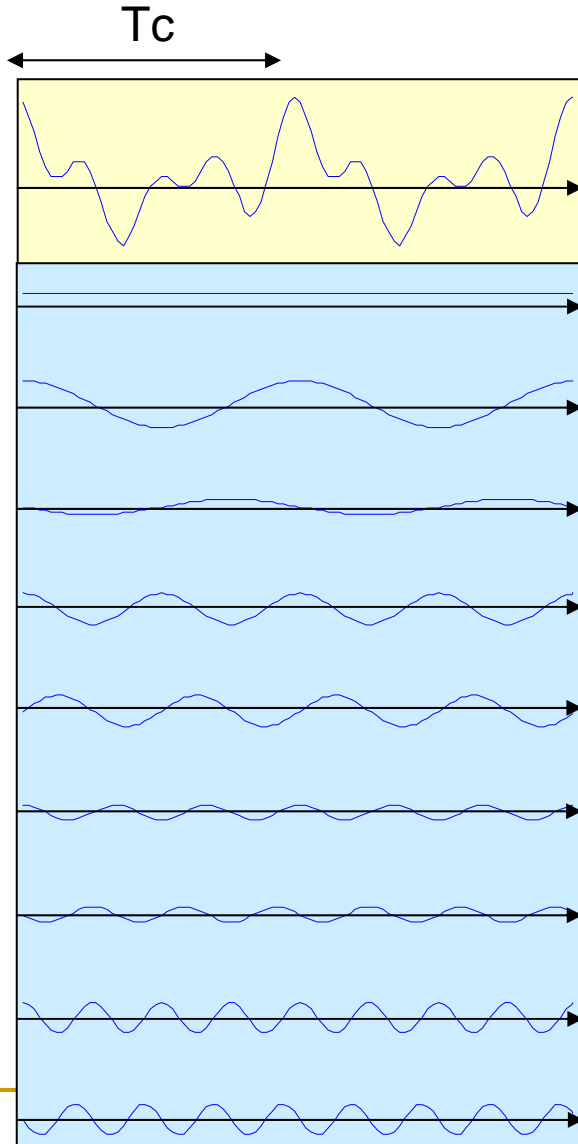


How we can divide $f(t)$ into sinusoids?



- Filter is used
- a_n and b_n

If we integrate in [0 to Tc]



$$\int_0^{T_c} f(t) dt$$

$$\int_0^{T_c} a_0 dt = T_c \cdot a_0$$

$$\int_0^{T_c} a_1 \cos(1\omega t) dt = 0$$

$$\int_0^{T_c} b_1 \sin(1\omega t) dt = 0$$

$$\int_0^{T_c} a_2 \cos(2\omega t) dt = 0$$

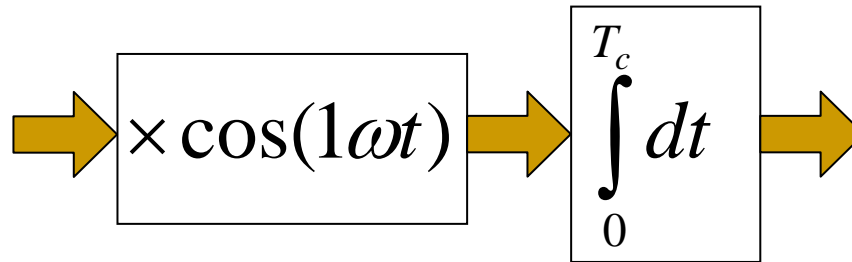
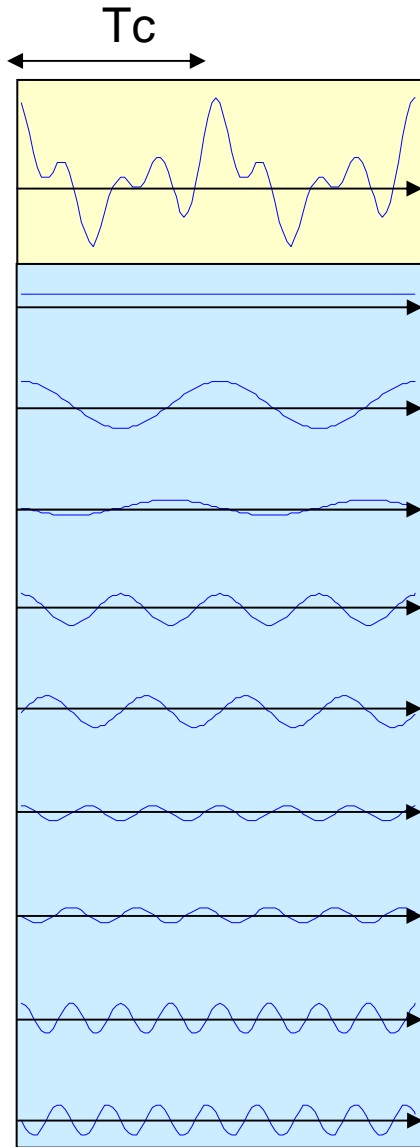
$$\int_0^{T_c} b_1 \sin(1\omega t) dt = 0$$

$$\int_0^{T_c} b_1 \sin(1\omega t) dt = 0$$

⋮

$$\int_0^{T_c} f(t) dt = T_c \cdot a_0$$

If we integrate in [0 to Tc] (2)



$$\int_0^{T_c} f(t) \times \cos(1\omega t) dt$$

$$a_1 \cdot \frac{T_c}{2}$$

0

0

0

0

0

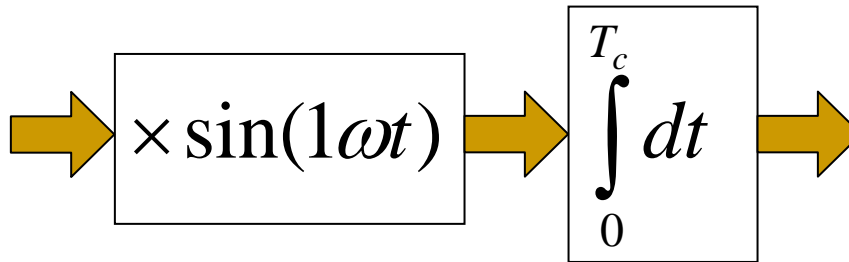
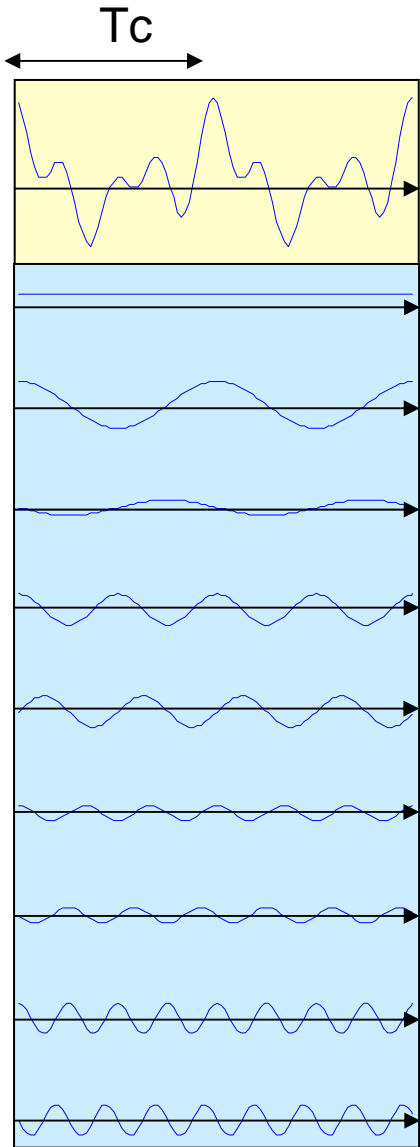
0

0

0

• a_1 can be computed

If we integrate in [0 to Tc] (3)

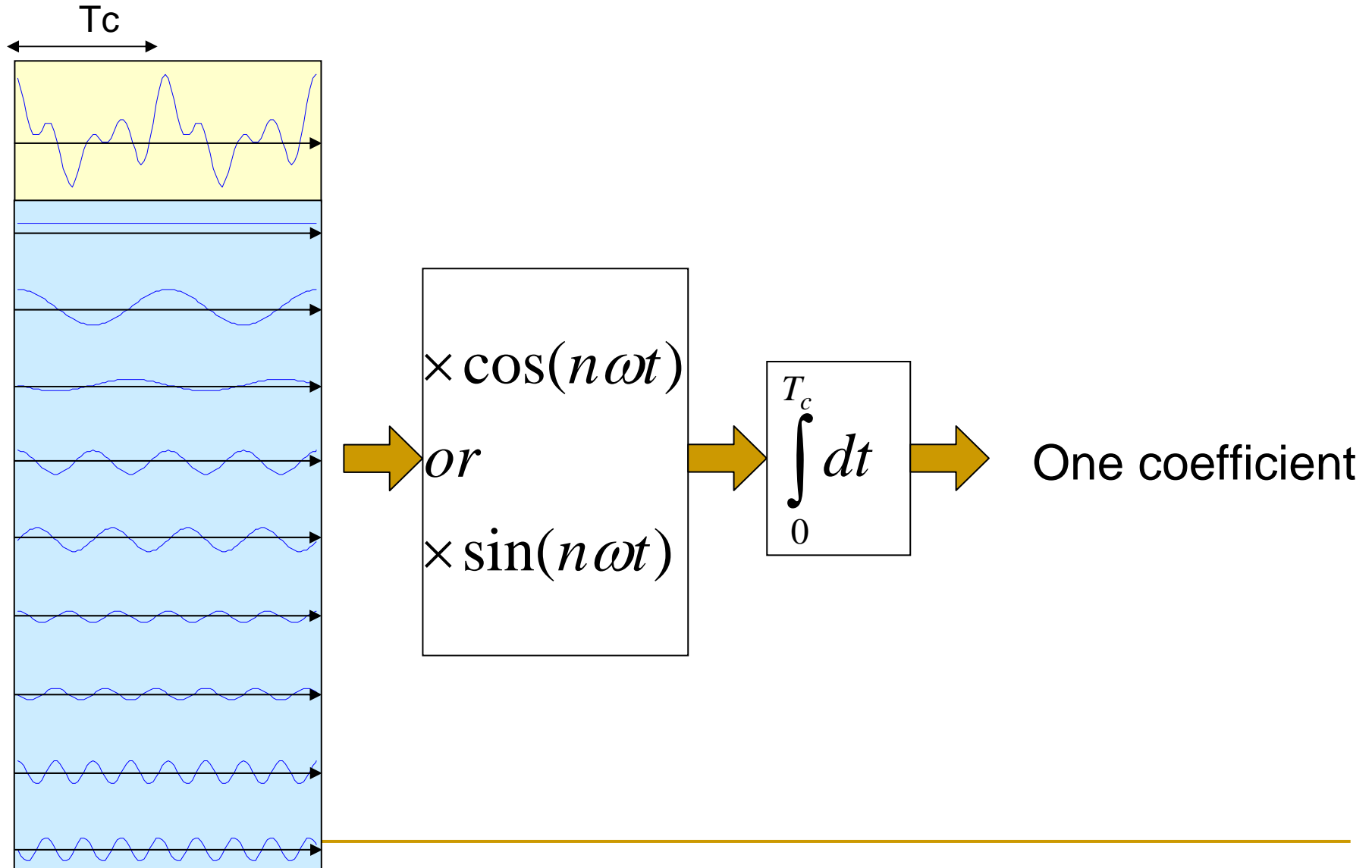


$$\int_0^{T_c} f(t) \times \cos(1\omega t) dt$$

$$\begin{array}{c}
 0 \\
 0 \\
 b_1 \cdot \frac{T_c}{2} \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
 \end{array}$$

• b_1 can be computed

By changing multiplier, each coefficient computed



Sinusoidal Orthogonality

- m, n : integer, $T_c = 1/f_0$

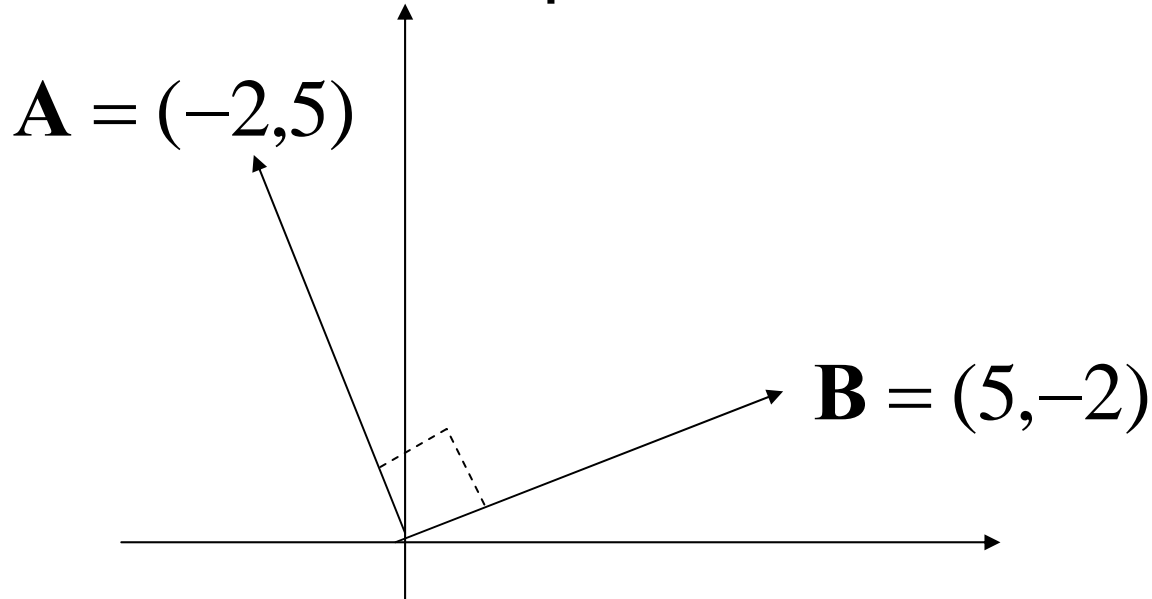
$$\int_0^{T_c} \cos(2\pi m f_0 t) \cdot \cos(2\pi n f_0 t) dt = \begin{cases} \frac{T_c}{2} & (m = n) \\ 0 & (m \neq n) \end{cases} \quad \longrightarrow \text{Orthogonal}$$

$$\int_0^{T_c} \sin(2\pi m f_0 t) \cdot \sin(2\pi n f_0 t) dt = \begin{cases} \frac{T_c}{2} & (m = n) \\ 0 & (m \neq n) \end{cases} \quad \longrightarrow \text{Orthogonal}$$

$$\int_0^{T_c} \cos(2\pi m f_0 t) \cdot \sin(2\pi n f_0 t) dt = 0 \quad \longrightarrow \text{Orthogonal}$$

Another Orthogonality (1)

- Vector inner product



$$\mathbf{A} \cdot \mathbf{B} = (-2) \times 5 + 5 \times 2 = 0$$

→ Orthogonal

$$= |\mathbf{A}| \cdot |\mathbf{B}| \cos \theta$$

= 90 degree

Another Orthogonality (2)

- n dimensional vector

$$\mathbf{A} = (a_1, a_2, \dots, a_n)$$

$$\mathbf{B} = (b_1, b_2, \dots, b_n)$$

$$\mathbf{A} \cdot \mathbf{B} = a_1 \cdot b_1 + a_2 \cdot b_2 + \dots + a_n \cdot b_n$$

IF $\mathbf{A} \cdot \mathbf{B} = 0$

THEN A and B are Orthogonal.

$\int_0^{T_c} dt$ is same as the N dim inner product

If signal is sampled,

$$\int_0^{T_c} \cos(2\pi m f_0 t) \cdot \cos(2\pi n f_0 t) dt$$

$$= \sum_{i=0}^{N-1} \cos(2\pi m f_0 \frac{i}{N} T_c) \cos(2\pi n f_0 \frac{i}{N} T_c)$$

$$= \sum_{i=0}^{N-1} \cos(2\pi m \frac{i}{N}) \cos(2\pi n \frac{i}{N})$$

$$= x_0 \cdot y_0 + x_1 \cdot y_1 + \dots + x_{N-1} \cdot y_{N-1}$$

Freq= n (Hz) sinusoids are Orthogonal each other (n =integer)

Fourier Series Summary

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t))$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n \frac{2\pi}{T_c} t) + b_n \sin(n \frac{2\pi}{T_c} t))$$

$$a_0 = \frac{1}{T_c} \int_0^{T_c} f(t) dt$$

$$a_n = \frac{2}{T_c} \int_0^{T_c} f(t) \cos(n\omega t) dt$$

$$b_n = \frac{2}{T_c} \int_0^{T_c} f(t) \sin(n\omega t) dt$$

Complex form Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{jn\omega t}$$

$$c_n = \frac{1}{T_c} \int_0^{T_c} f(t) e^{-jn\omega t} dt$$

$$\int_0^{T_c} e^{jn\omega t} e^{-jm\omega t} dt$$

$$= \int_0^{T_c} e^{j\omega(n-m)t} dt = \begin{cases} T_c & (m = n) \\ 0 & (m \neq n) \end{cases} \rightarrow \text{Orthogonal}$$

HW2

$$f(t) = \begin{cases} 1 & 0 \leq t \leq \frac{1}{2}T_0 \\ -1 & \frac{1}{2}T_0 \leq t < T_0 \end{cases}$$

- [2-1] Compute the complex form Fourier Series coefficient c_n for $f(x)$.
- [2-2] Draw the Spectrum of $f(t)$ when $T_0=0.04\text{sec}$.

2.30