OUTLINE

- Periodic Signal
- Fourier series introduction
- Sinusoids Orthogonality
- Integration vs inner product

Consider any wave is sum of simple sin and cosine

$$f(t)$$

$$a_0 = 1$$

$$a_1 = 3\cos(\omega t)$$

$$b_1 = -1\sin(\omega t)$$

$$a_2 = 2\cos(2\omega t)$$

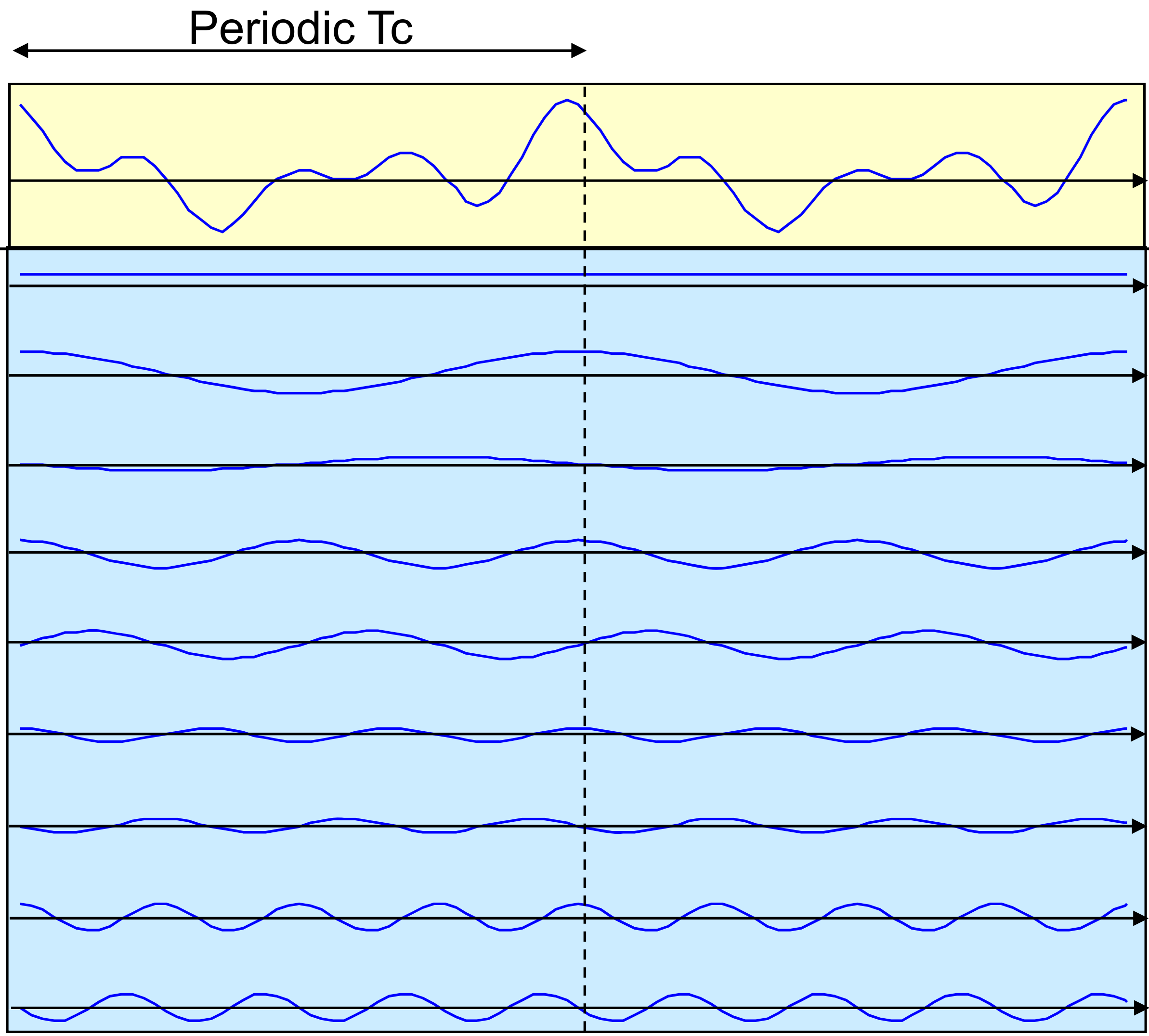
$$b_2 = 2\sin(2\omega t)$$

$$a_3 = 1\cos(3\omega t)$$

$$b_3 = 1\sin(3\omega t)$$

$$a_4 = 2\cos(4\omega t)$$

$$b_4 = -2\sin(4\omega t)$$



Periodic signal is composed of DC + same frequency sinusoid + multiple frequency sinusoids

$$\omega = 2\pi f_c = 2\pi \frac{1}{T_c}$$

$$f(t) =$$

$$1$$ **Frequency = 0 Hz**

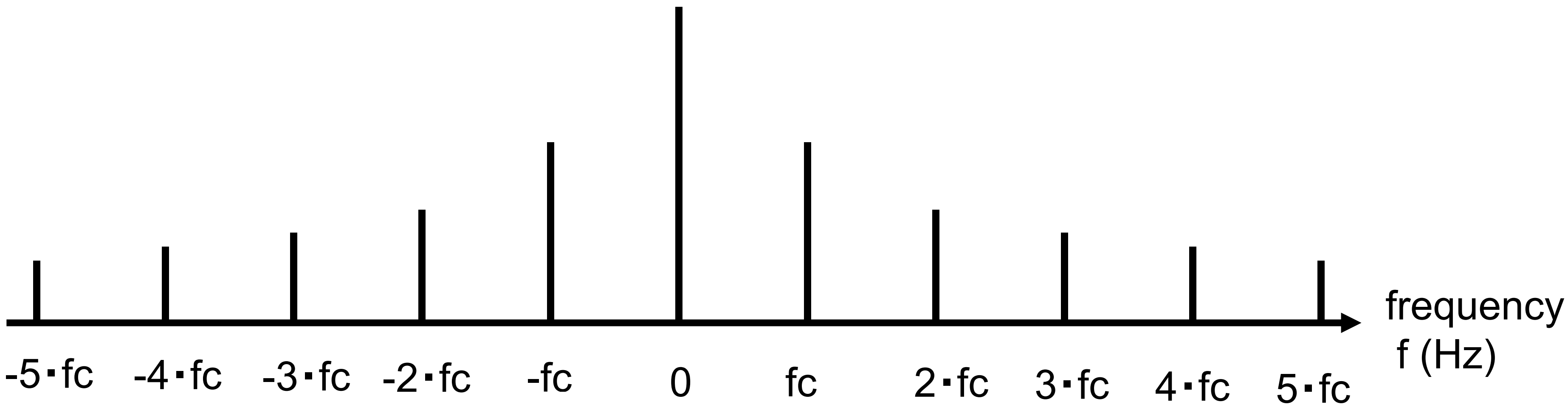
$$+3\cos(\omega t) - 1\sin(\omega t)$$ **Basic frequency fc=1/Tc**

$$+2\cos(2\omega t) + 2\sin(2\omega t)$$ **2 x fc**

$$+1\cos(3\omega t) + 1\sin(3\omega t)$$ **3 x fc**

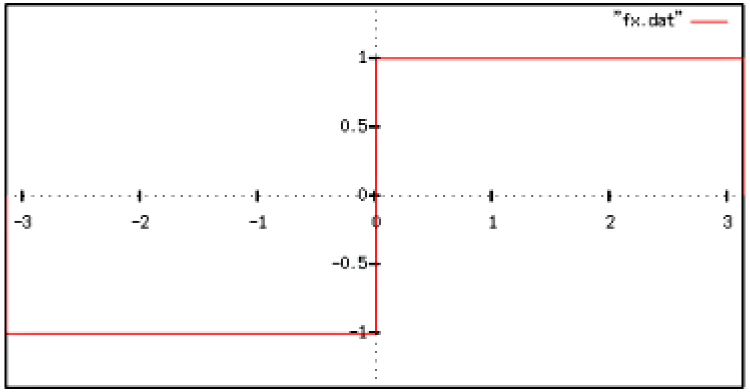
$$+2\cos(4\omega t) - 2\sin(4\omega t)$$ **4 x fc**

Spectrum of periodic signal



There are only n * fc (n=integer) frequencies!

Another example (even rectangular pulse)

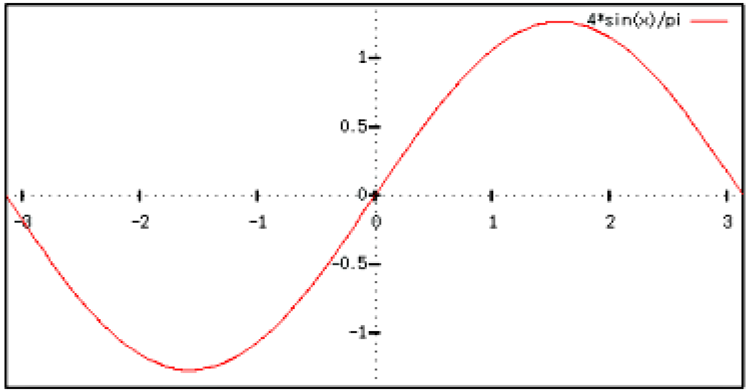


$$f(x) = \begin{cases} -1 & (-\pi < x < 0) \\ 1 & (0 < x < \pi) \end{cases}, \qquad f(x+2\pi) = f(x)$$

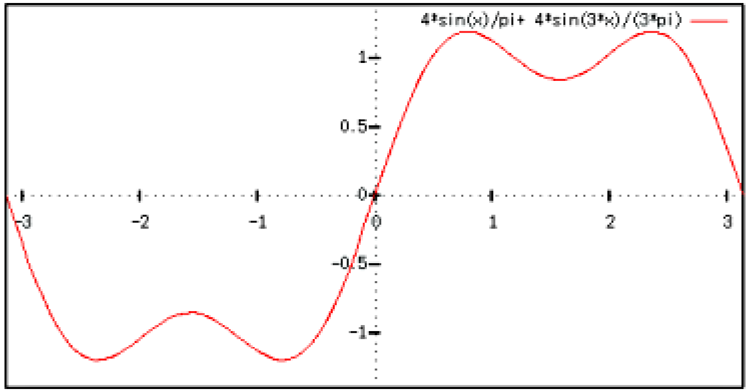
$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots\right)$$
$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\{(2n-1)x\}}{2n-1}$$

Increase the number of sum (1)

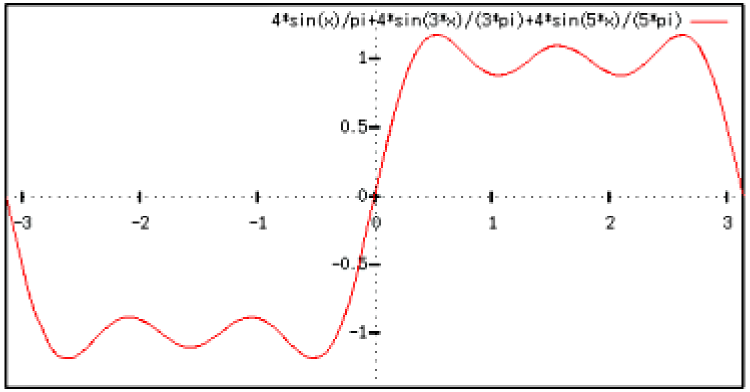
N=1



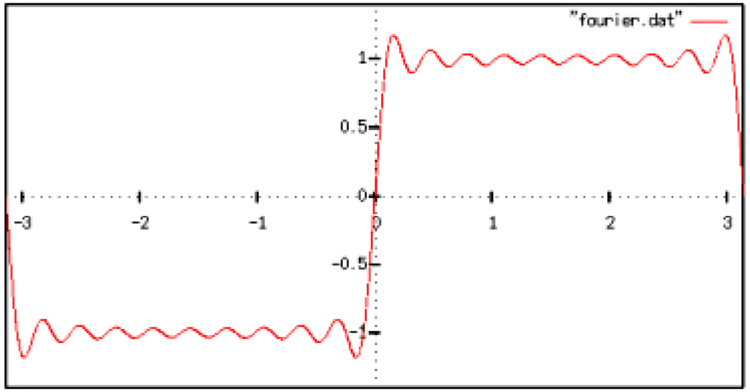
N=2



N=3

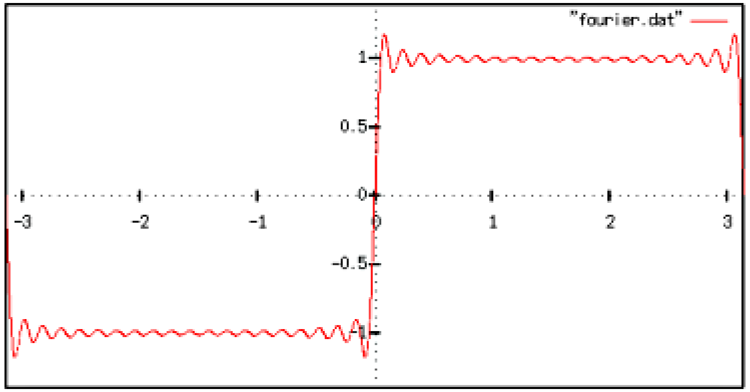


N=10

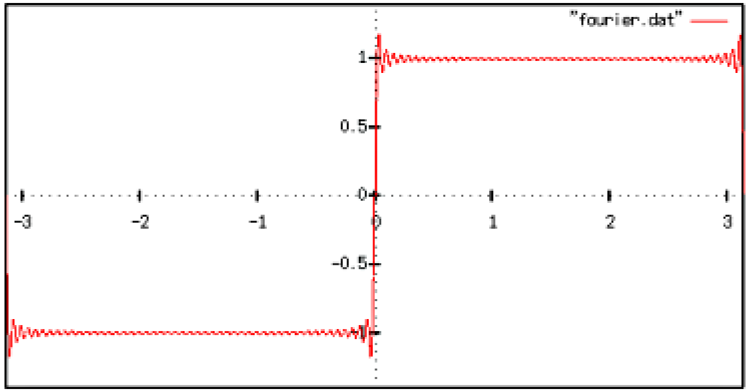


Increase the number of sum (2)

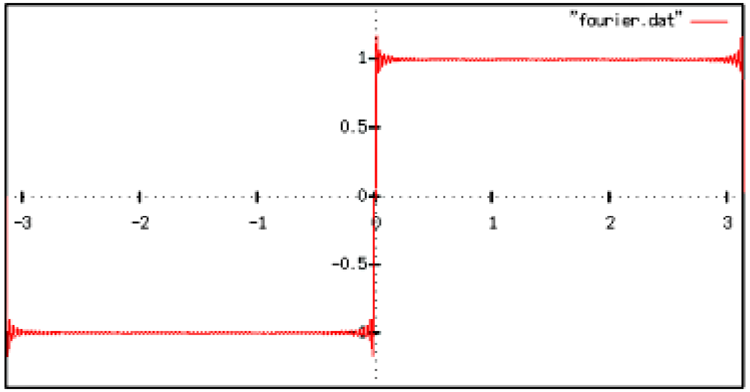
N=20



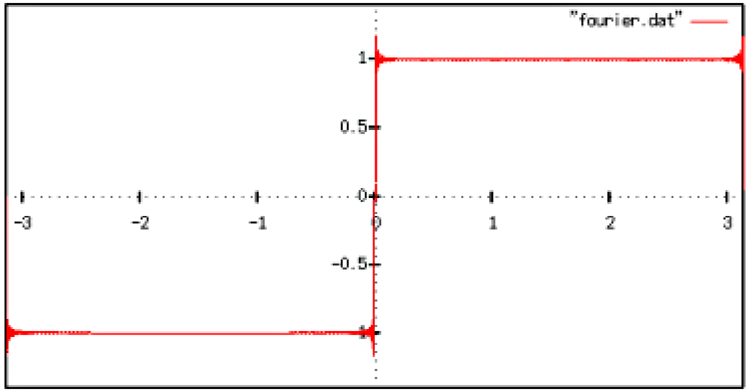
N=50



N=100



N=200



Fourier

- Jean Baptiste Joseph,
  Baron de Frourier
  France, 1778/Mar/21 – 1830/May/16
- Fourier Series paper is written in 1807
- Even discontinue function (such as rectangular pulse) can be composed of many sinusoids.
- Nobody believed the paper at that time.

Fourier Series

- If f(t) 's period is Tc… $\omega = 2\pi f_c = 2\pi \frac{1}{T_c}$

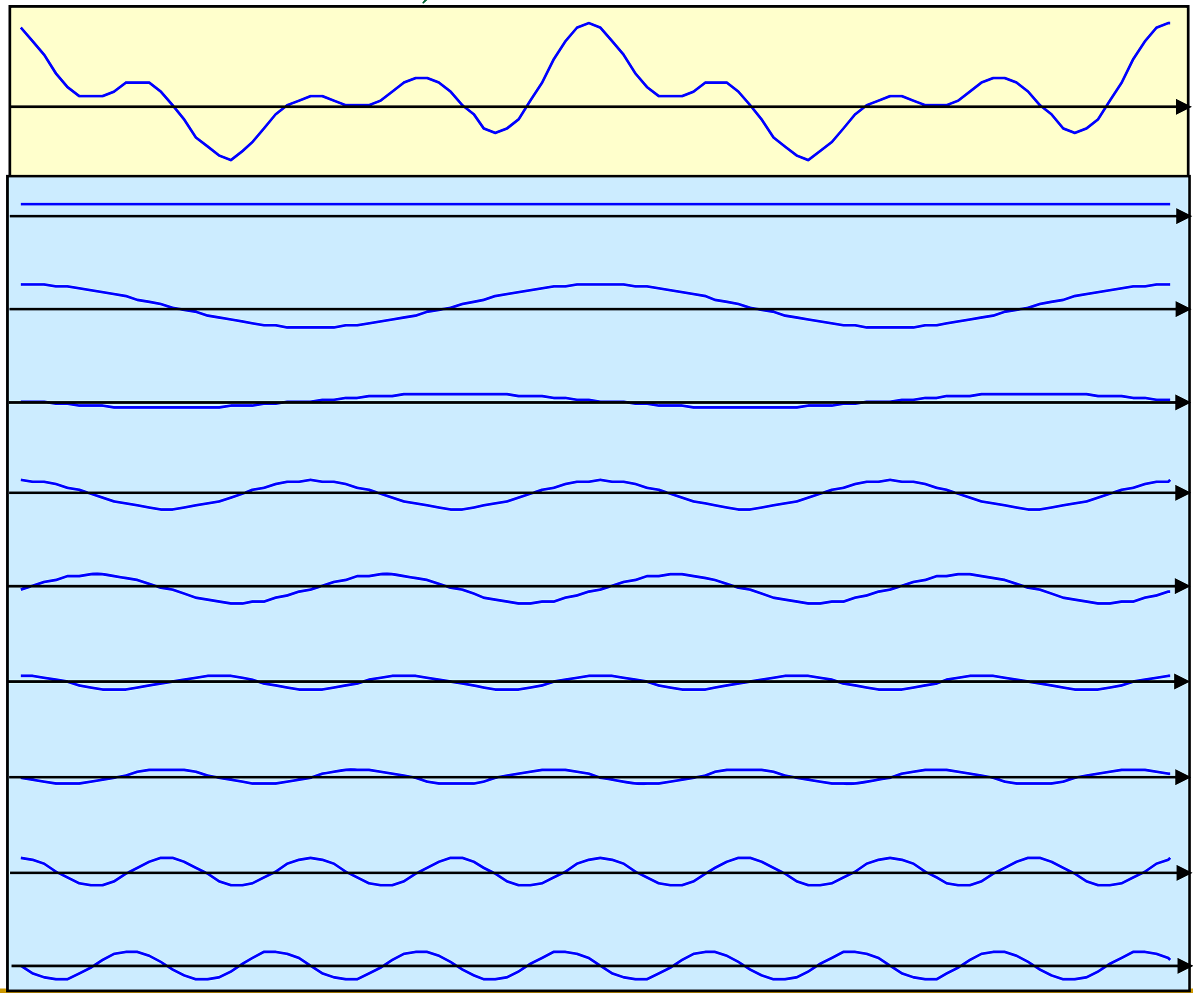
$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t))$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n \frac{2\pi}{T_c} t) + b_n \sin(n \frac{2\pi}{T_c} t))$$

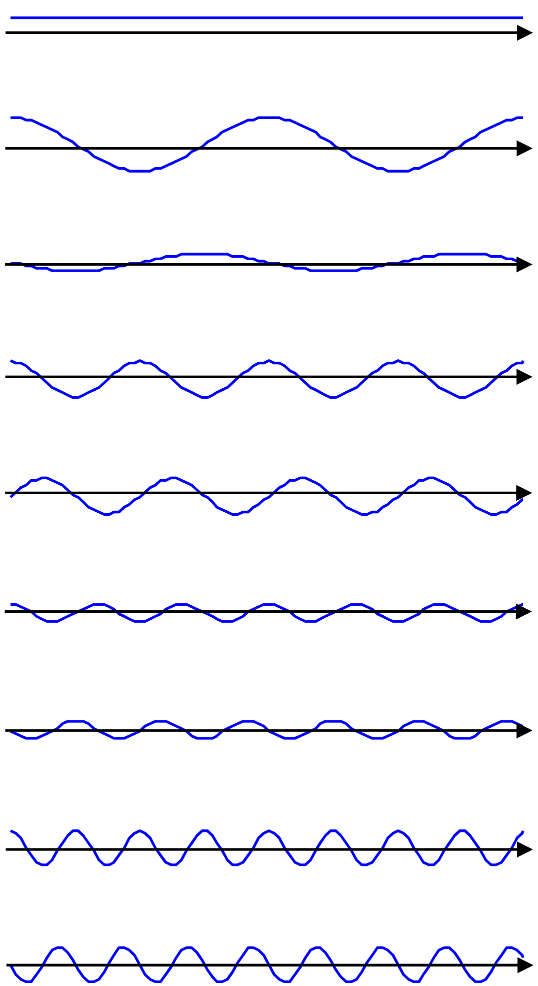
- If we use complex exponential…,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{jn\omega t}$$

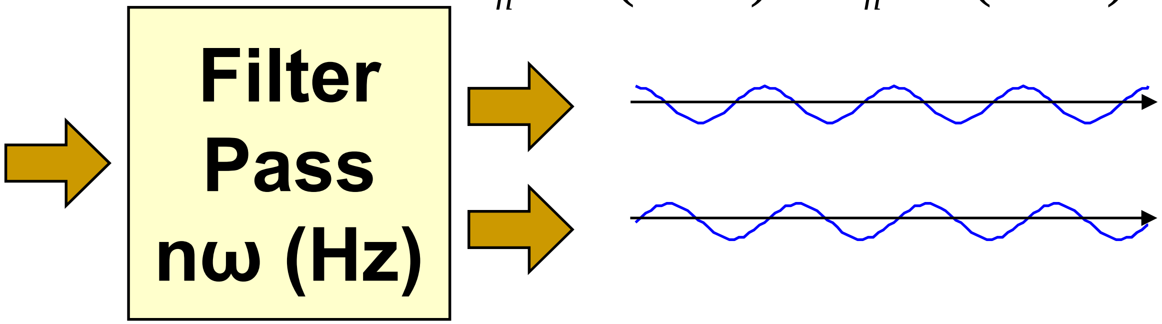
Anyway, when you see the periodic signal, Please think it is just sum of sinusoids!!!



How we can divide f(t) into sinusoids?

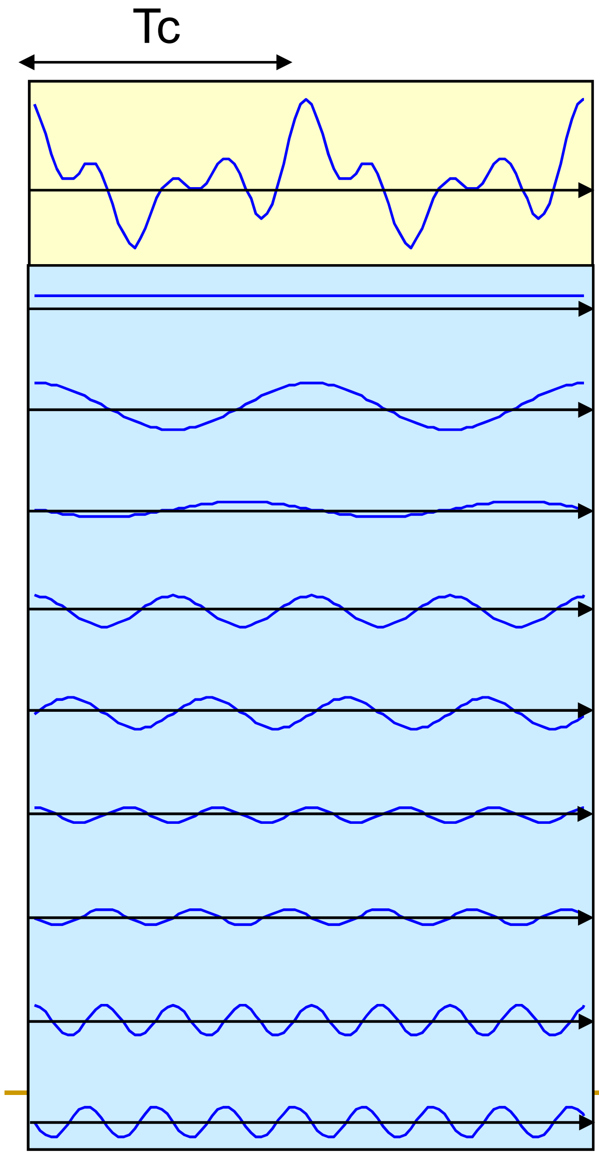


$$a_n \cos(n\omega t) + b_n \sin(n\omega t)$$



- Filter is used
- a_n and b_n

If we integrate in [0 to Tc]



$$\int_0^{T_c} f(t)dt$$

$$\int_0^{Tc} a_0 dt = T_c \cdot a_0$$

$$\int_0^{Tc} a_1 \cos(1\omega t)dt = 0$$

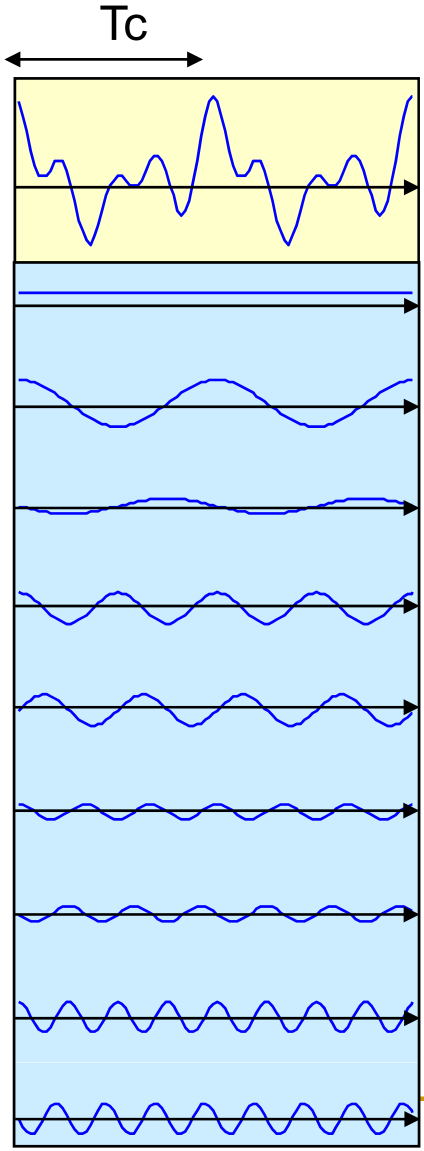
$$\int_0^{Tc} b_1 \sin(1\omega t)dt = 0$$

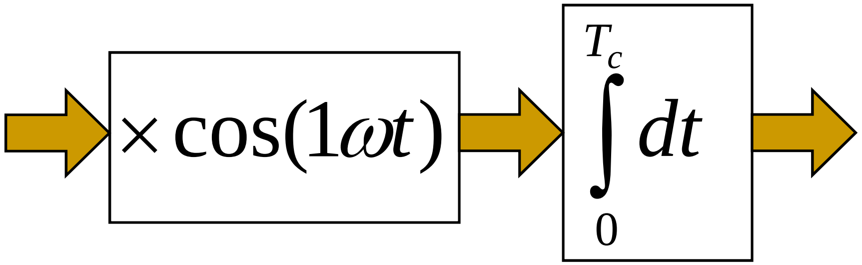
$$\int_0^{Tc} a_2 \cos(2\omega t)dt = 0$$

$$\int_0^{Tc} b_1 \sin(1\omega t)dt = 0$$

$$\int_0^{T_c} f(t)dt = T_c \cdot a_0$$

If we integrate in [0 to Tc] (2)





$$\int_0^{T_c} f(t) \times \cos(1\omega t) dt$$

0

$a_1 \cdot \frac{T_c}{2}$

0

0

0

0

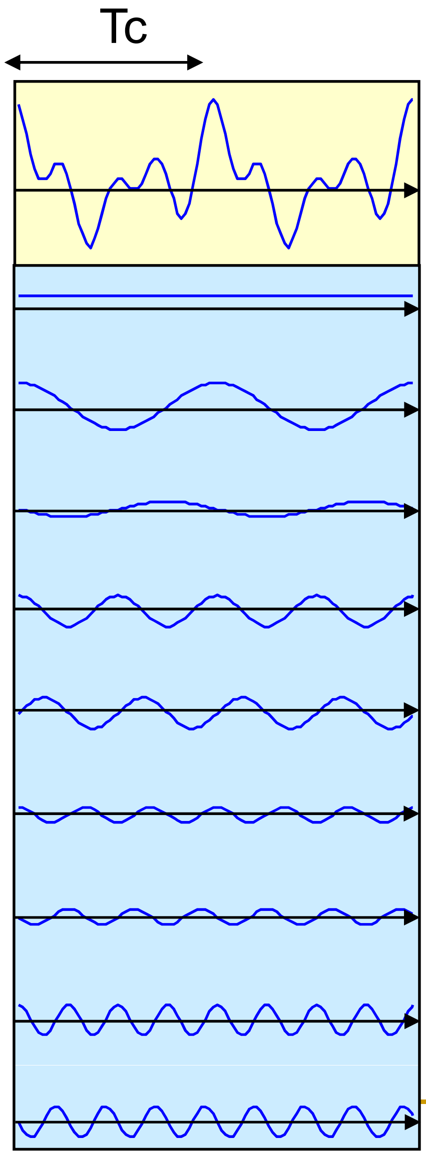
0

0

0

- a_1 can be computed

If we integrate in [0 to Tc] (3)



$$\times \sin(1\omega t) \rightarrow \int_0^{T_c} dt \rightarrow$$

$$\int_0^{T_c} f(t) \times \cos(1\omega t) dt$$

0

0

$b_1 \cdot \frac{T_c}{2}$

0

0

0

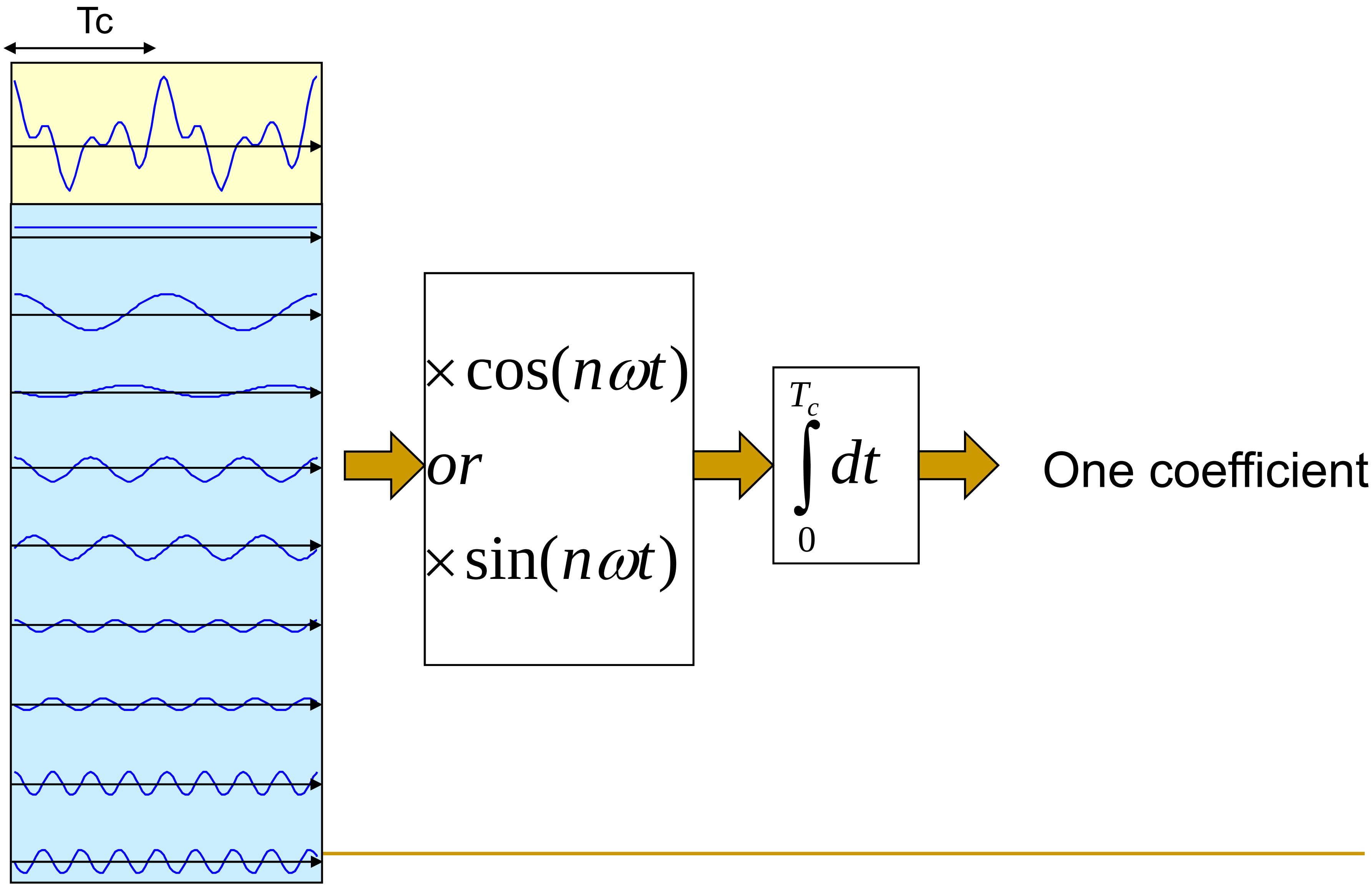
0

0

0

- b_1 can be computed

By changing multiplier, each coefficient computed



Sinusoidal Orthogonality

- m,n: integer, $T_c=1/f_0$

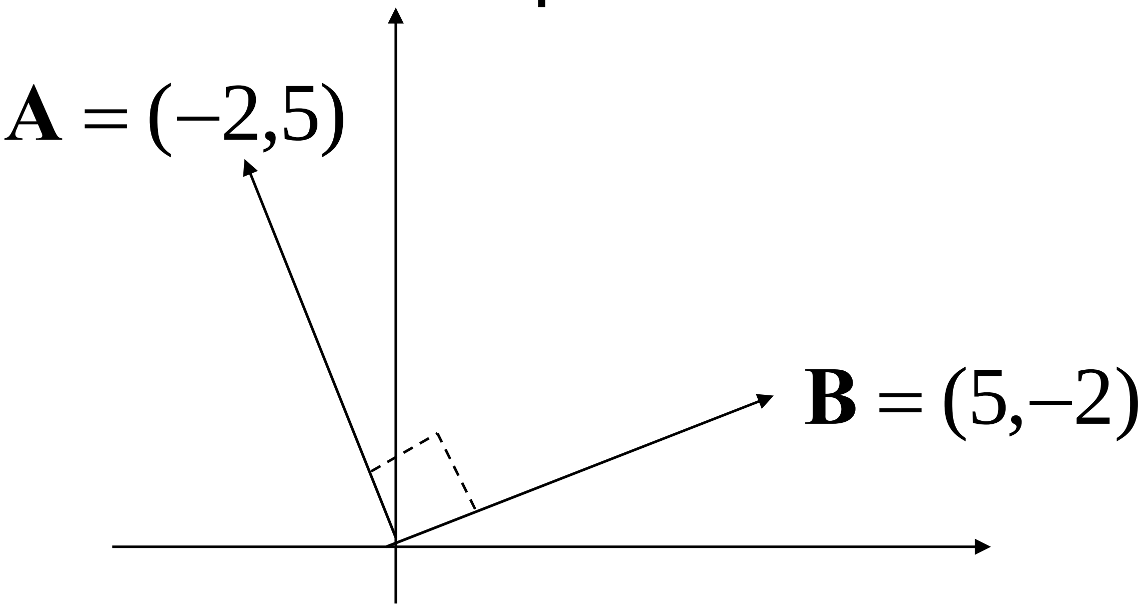
$$\int_0^{T_c} \cos(2\pi m f_0 t)\cdot\cos(2\pi n f_0 t)dt = \begin{cases} \frac{T_c}{2} (m=n) \\ 0\,(m \neq n) \end{cases}$$ → Orthogonal

$$\int_0^{T_c} \sin(2\pi m f_0 t)\cdot\sin(2\pi n f_0 t)dt = \begin{cases} \frac{T_c}{2} (m=n) \\ 0\,(m \neq n) \end{cases}$$ → Orthogonal

$$\int_0^{T_c} \cos(2\pi m f_0 t)\cdot\sin(2\pi n f_0 t)dt = 0$$ → Orthogonal

Another Orthogonality (1)

- Vector inner product



$$\mathbf{A} \cdot \mathbf{B} = (-2) \times 5 + 5 \times 2 = 0$$

→ Orthogonal

$$= |\mathbf{A}| \cdot |\mathbf{B}| \cos\theta$$

Θ=90 degree

Another Orthogonality (2)

- n dimensional vector

$$\mathbf{A} = (a_1, a_2, \cdots, a_n)$$

$$\mathbf{B} = (b_1, b_2, \cdots, a_n)$$

$$\mathbf{A} \cdot \mathbf{B} = a_1 \cdot b_1 + a_2 \cdot b_2 + \cdots + a_n \cdot b_n$$

IF $\mathbf{A} \cdot \mathbf{B} = 0$

THEN A and B are Orthogonal.

$\int_0^{T_c} dt$ is same as the N dim inner product

$$If \quad signal \quad is \quad sampled,$$

$$\int_0^{T_c} \cos(2\pi m f_0 t) \cdot \cos(2\pi n f_0 t) dt$$

$$= \sum_{i=0}^{N-1} \cos(2\pi m f_0 \frac{i}{N} T_c) \cos(2\pi n f_0 \frac{i}{N} T)$$

$$= \sum_{i=0}^{N-1} \cos(2\pi m \frac{i}{N}) \cos(2\pi n \frac{i}{N})$$

$$= x_0 \cdot y_0 + x_1 \cdot y_1 + \cdots + x_{N-1} \cdot y_{N-1}$$

Freq=nω(Hz) sinusoids are Orthogonal each other (n=integer)

Fourier Series Summary

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\omega t) + b_n \sin(n\omega t))$$

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\frac{2\pi}{T_c}t) + b_n \sin(n\frac{2\pi}{T_c}t))$$

$$a_0 = \frac{1}{T_c}\int_0^{T_c} f(t)dt$$

$$a_n = \frac{2}{T_c}\int_0^{T_c} f(t)\cos(n\omega t)dt$$

$$b_n = \frac{2}{T_c}\int_0^{T_c} f(t)\sin(n\omega t)dt$$

Complex form Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{jn\omega t}$$

$$c_n = \frac{1}{T_c}\int_0^{T_c} f(t)e^{-jn\omega t}dt$$

$$\int_0^{T_c} e^{jn\omega t}e^{-jm\omega t}dt$$

$$= \int_0^{T_c} e^{j\omega(n-m)t}dt = \begin{cases} T_c (m = n) \\ 0 (m \neq n) \end{cases}$$

$\longrightarrow$ Orthogonal

HW2

$$f(t)=\begin{cases}1 & 0\le t\le \frac{1}{2}T_0\\ -1 & \frac{1}{2}T_0\le t<T_0\end{cases}$$

- [2-1]Compute the complex form Fourier Series coefficient cn for f(x).
- [2-2]Draw the Spectrum of f(t) when T_0=0.04sec.

2.30