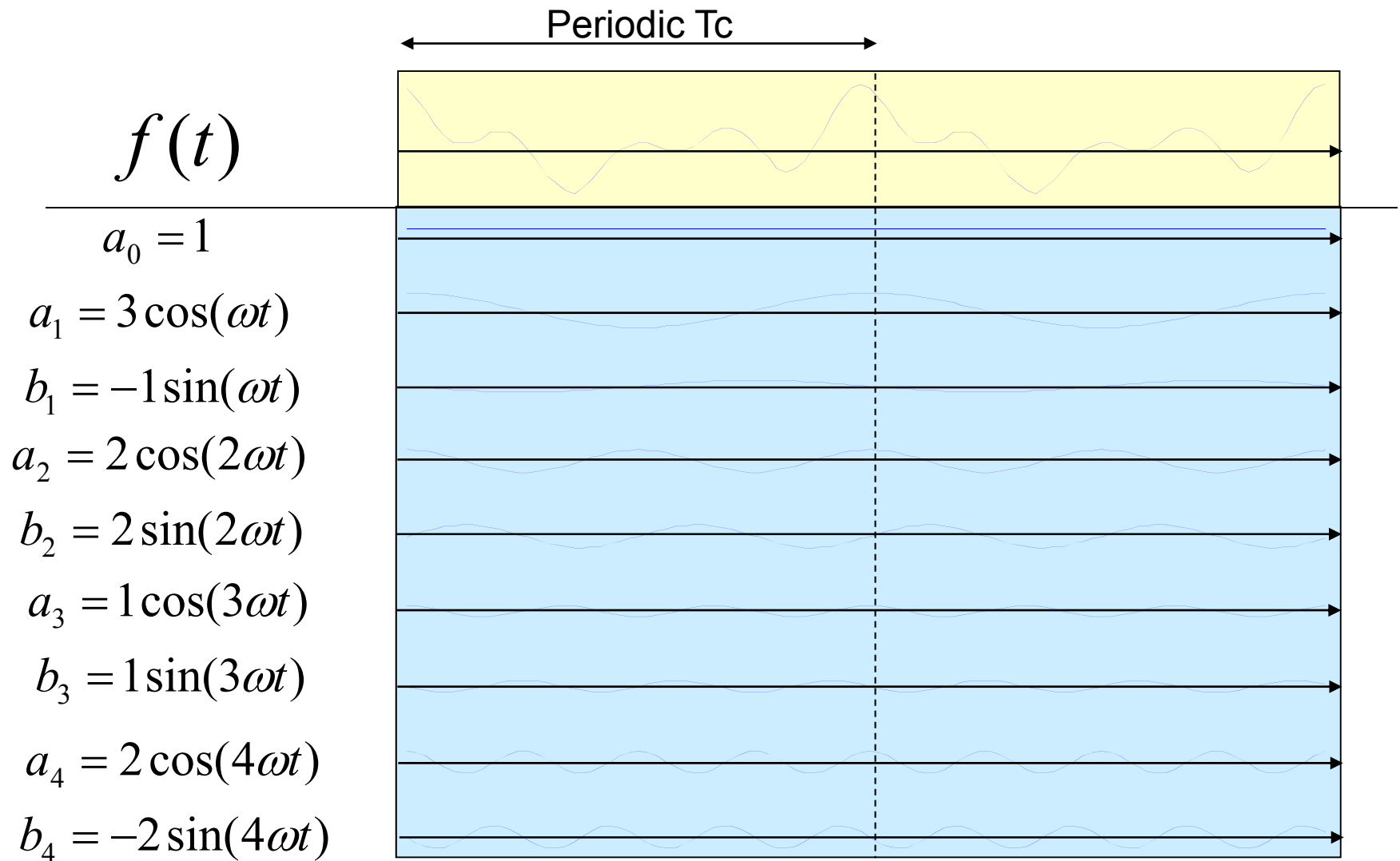


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# OUTLINE

- Periodic Signal
- Fourier series introduction
- Sinusoids Orthogonality
- Integration vs inner product

# Consider any wave is sum of simple sin and cosine



# Periodic signal is composed of DC + same frequency sinusoid + multiple frequency sinusoids

$$f(t) =$$

1

$$+ 3 \cos(\omega t) - 1 \sin(\omega t)$$

$$+ 2 \cos(2\omega t) + 2 \sin(2\omega t)$$

$$+ 1 \cos(3\omega t) + 1 \sin(3\omega t)$$

$$+ 2 \cos(4\omega t) - 2 \sin(4\omega t)$$

$$\omega = 2\pi f_c = 2\pi \frac{1}{T_c}$$

**Frequency = 0 Hz**

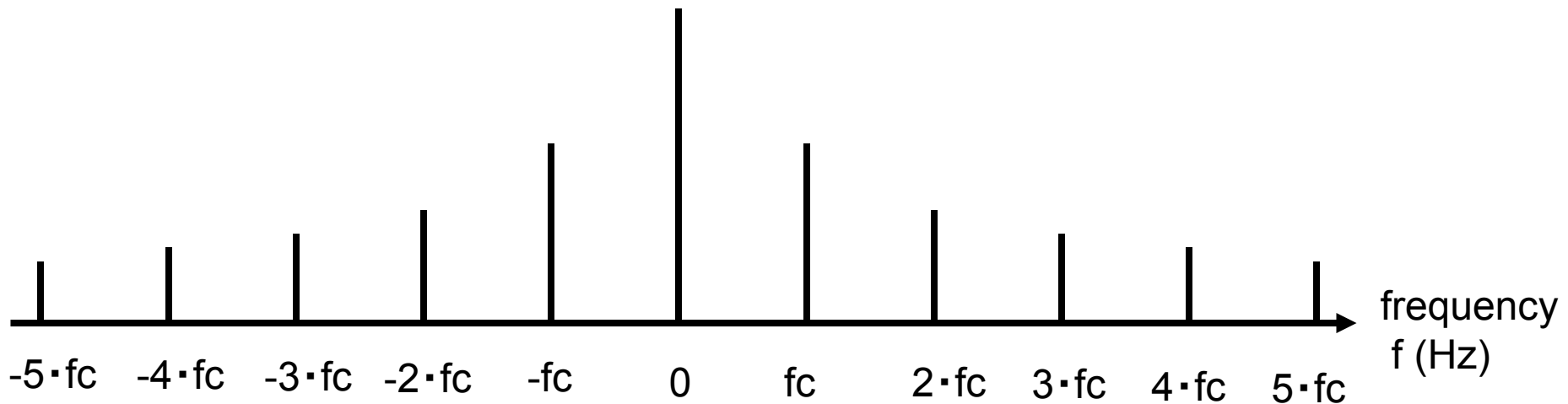
**Basic frequency  $f_c=1/T_c$**

**2 x  $f_c$**

**3 x  $f_c$**

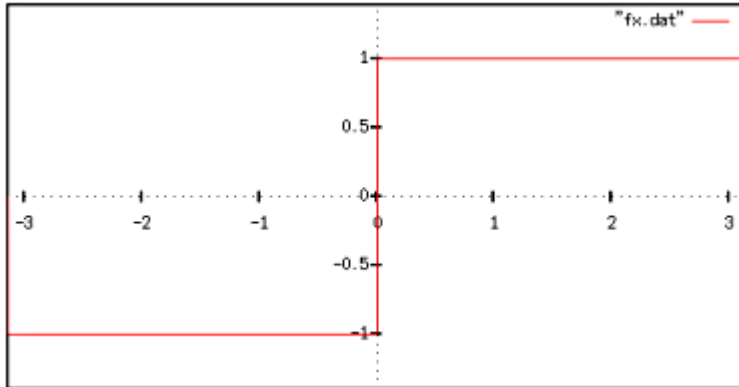
**4 x  $f_c$**

# Spectrum of periodic signal



There are only  $n * fc$  ( $n=\text{integer}$ ) frequencies!

# Another example (even rectangular pulse)

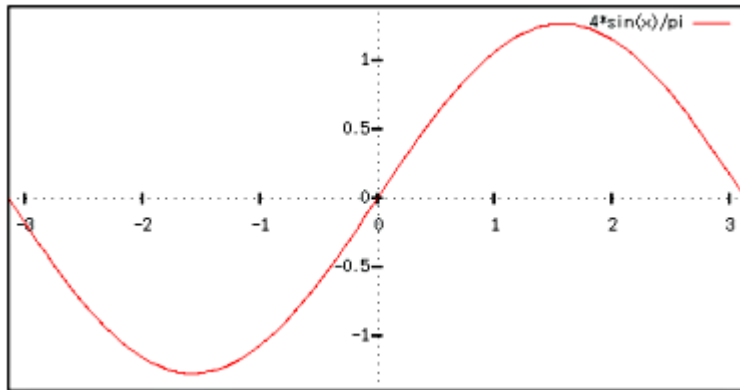


$$f(x) = \begin{cases} -1 & (-\pi < x < 0) \\ 1 & (0 < x < \pi) \end{cases}, \quad f(x + 2\pi) = f(x)$$

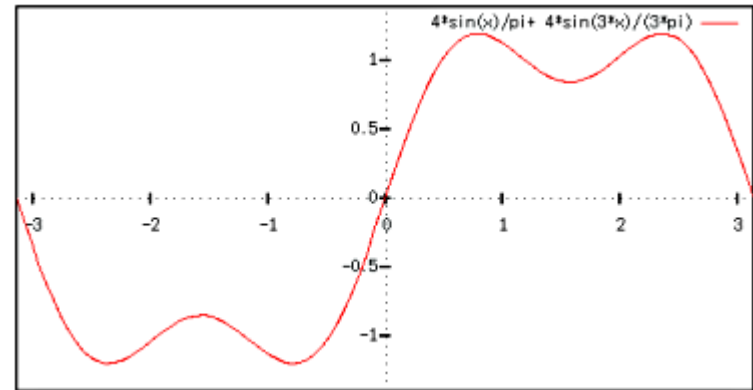
$$\begin{aligned} f(x) &= \frac{4}{\pi} \left( \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \frac{1}{7} \sin 7x + \dots \right) \\ &= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin\{(2n-1)x\}}{2n-1} \end{aligned}$$

# Increase the number of sum (1)

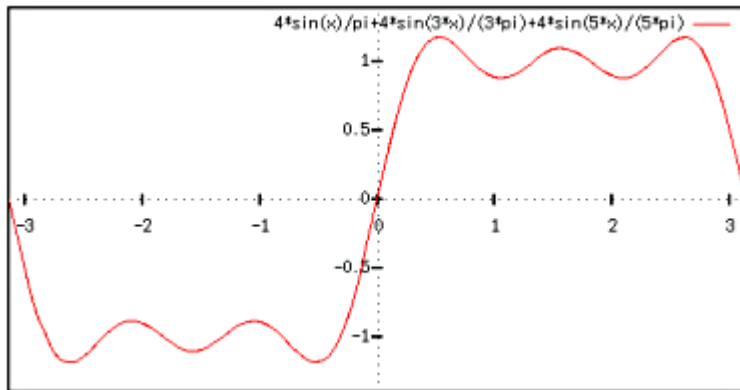
N=1



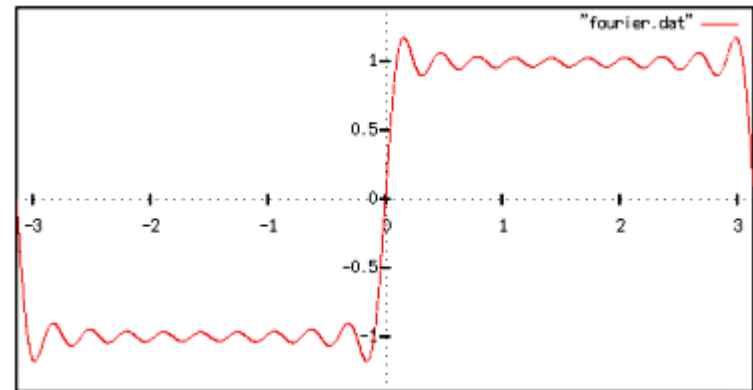
N=2



N=3

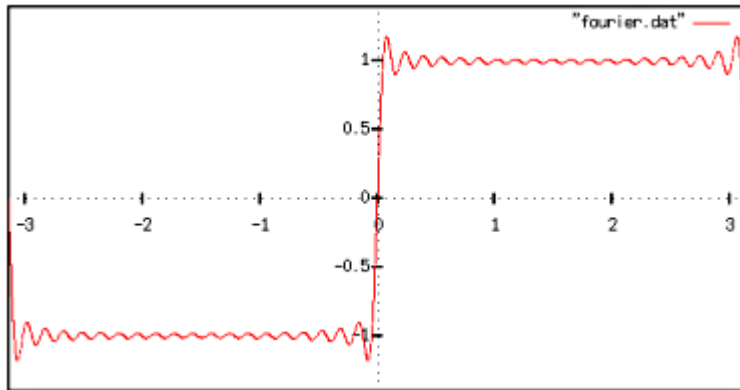


N=10

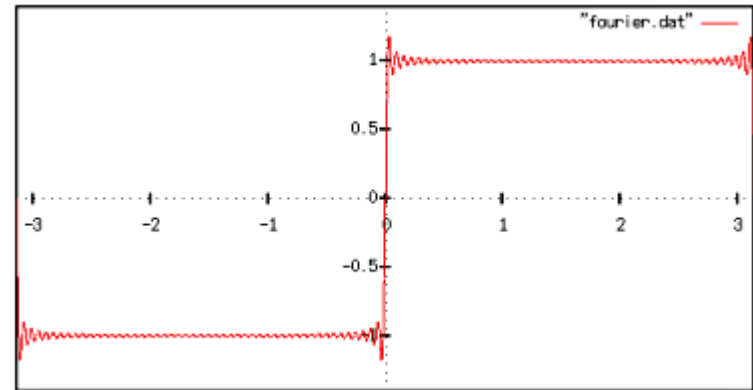


# Increase the number of sum (2)

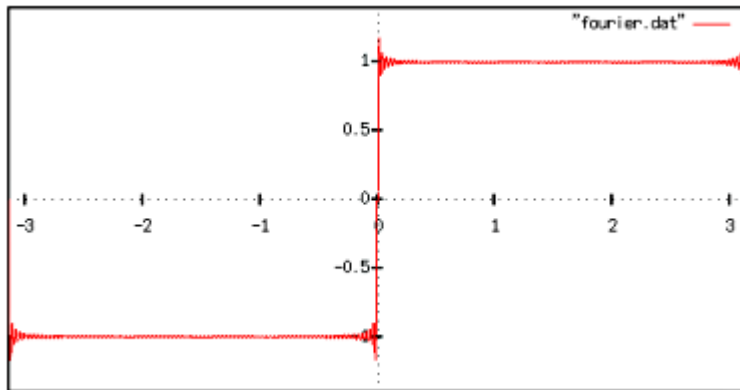
N=20



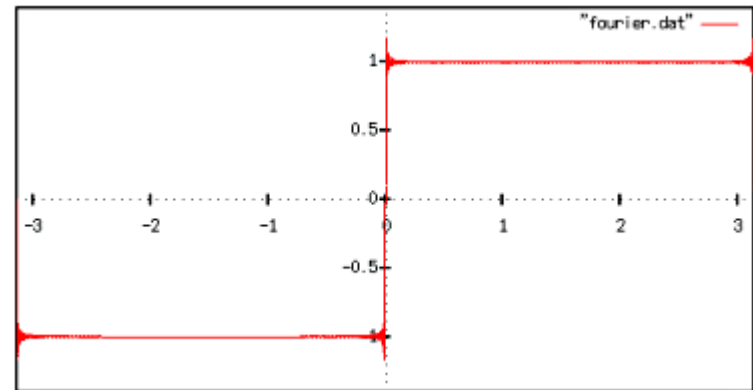
N=50



N=100



N=200



# Fourier

- Jean Baptiste Joseph, Baron de Frourier  
France, 1778/Mar/21 – 1830/May/16
- Fourier Series paper is written in 1807
- Even discontinue function (such as rectangular pulse) can be composed of many sinusoids.
- Nobody believed the paper at that time.





# Fourier Series

- If  $f(t)$  's period is  $T_c$ ...  $\omega = 2\pi f_c = 2\pi \frac{1}{T_c}$

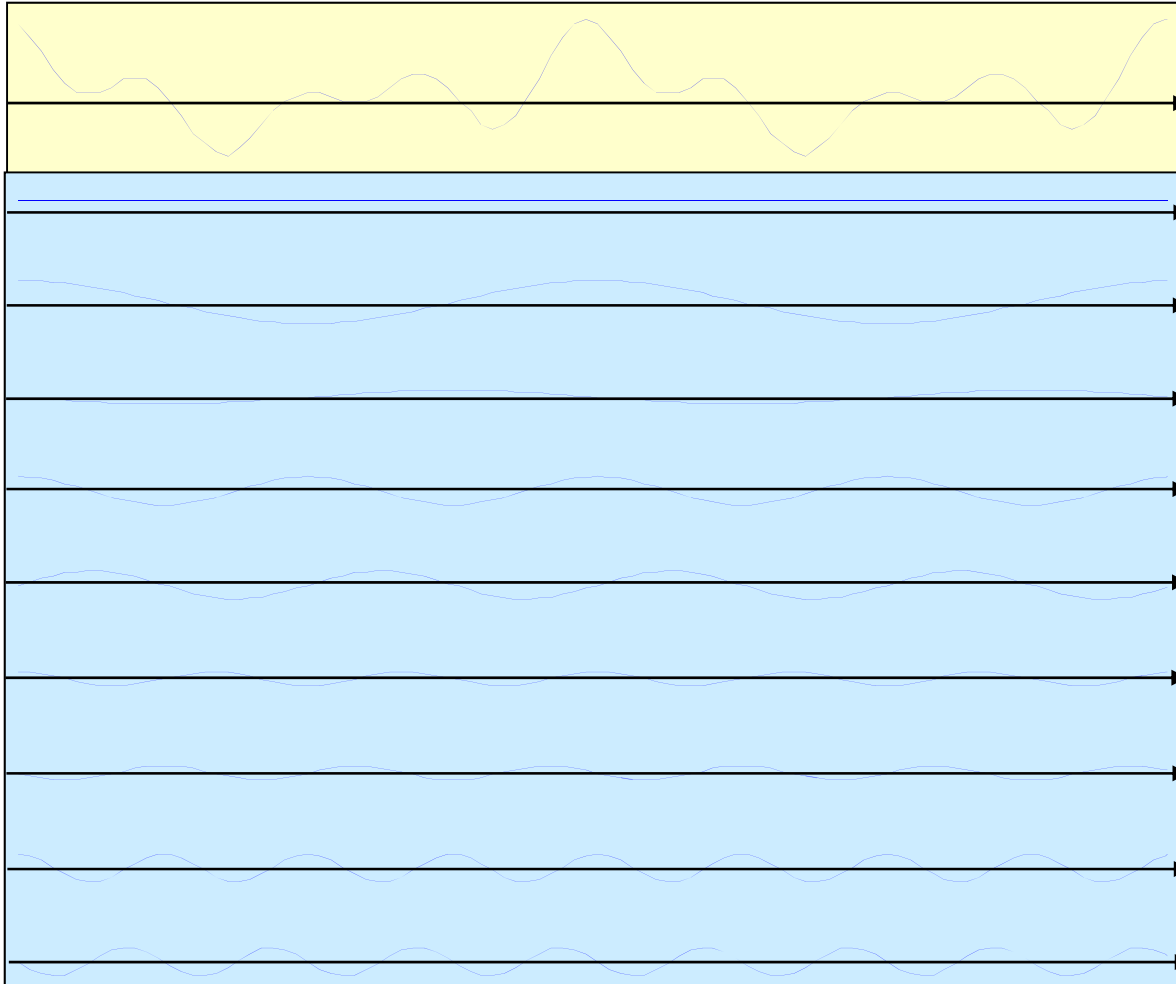
$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t))$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n \frac{2\pi}{T_c} t) + b_n \sin(n \frac{2\pi}{T_c} t))$$

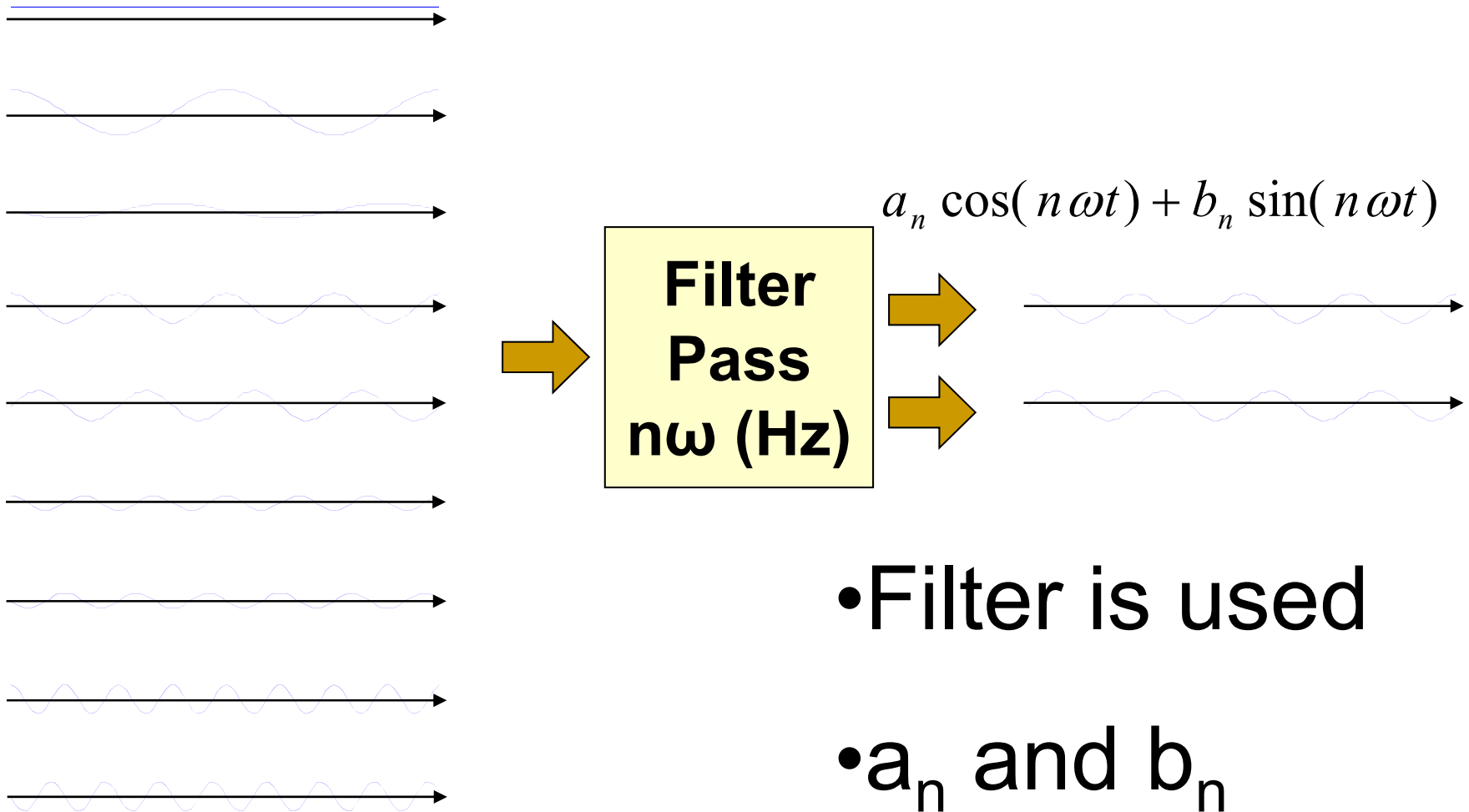
- If we use complex exponential....,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{jn\omega t}$$

Anyway, when you see the periodic signal,  
Please think it is just sum of sinusoids!!!

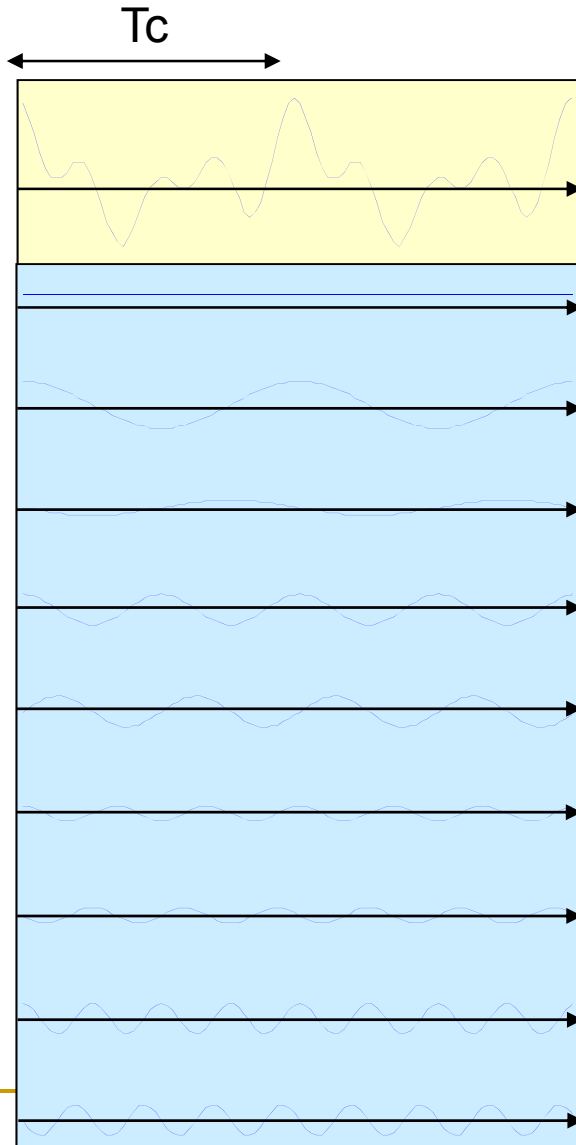


# How we can divide $f(t)$ into sinusoids?



- Filter is used
- $a_n$  and  $b_n$

# If we integrate in [ 0 to Tc ]



$$\int_0^{T_c} f(t) dt$$

$$\int_0^{T_c} a_0 dt = T_c \cdot a_0$$

$$\int_0^{T_c} a_1 \cos(\omega t) dt = 0$$

$$\int_0^{T_c} b_1 \sin(\omega t) dt = 0$$

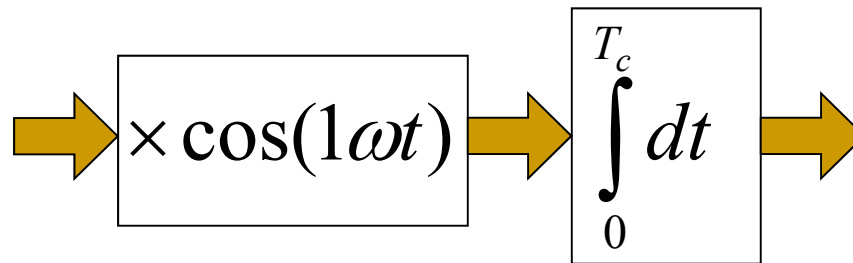
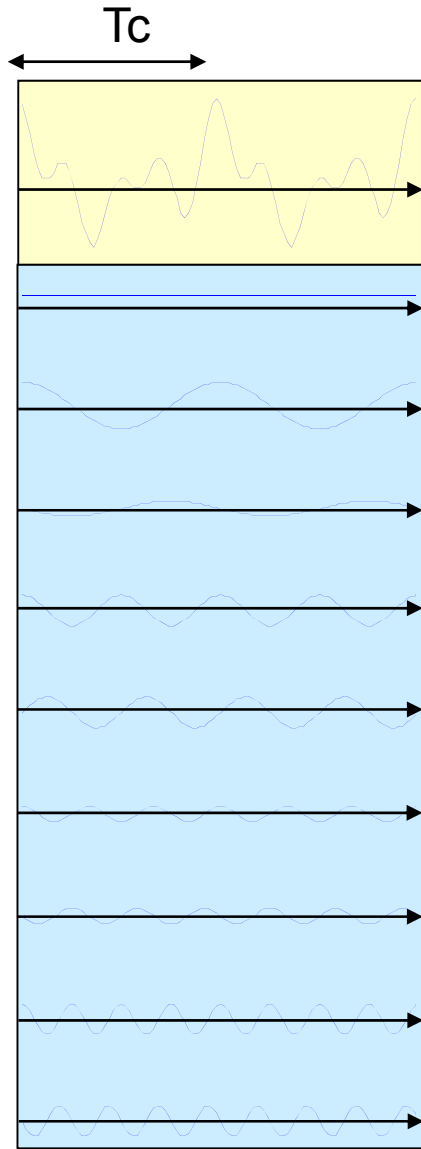
$$\int_0^{T_c} a_2 \cos(2\omega t) dt = 0$$

$$\int_0^{T_c} b_1 \sin(1\omega t) dt = 0$$

⋮

$$\int_0^{T_c} f(t) dt = T_c \cdot a_0$$

# If we integrate in [ 0 to Tc ] (2)



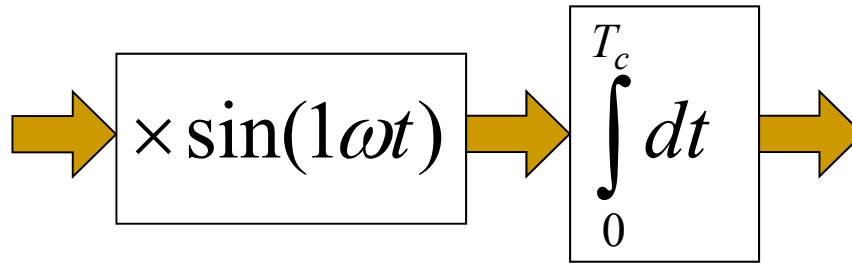
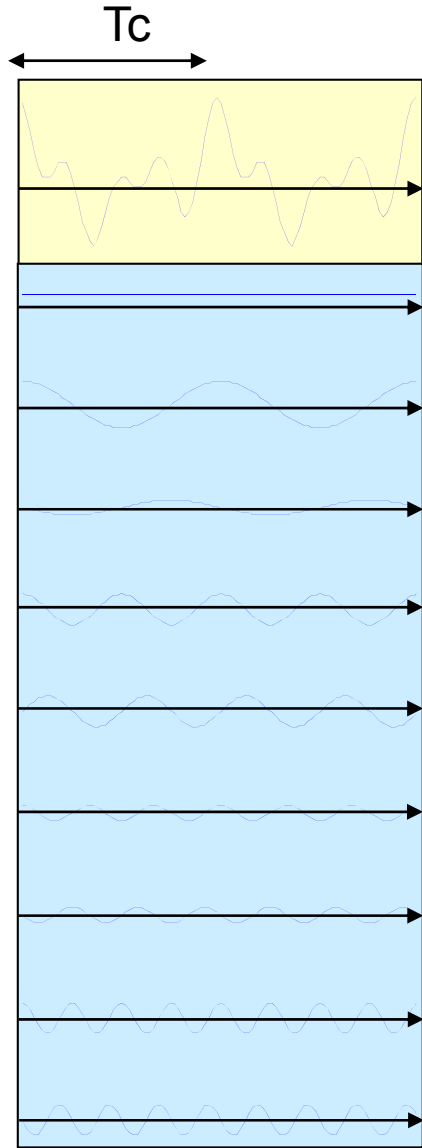
$$\int_0^{T_c} f(t) \times \cos(1\omega t) dt$$

$$a_1 \cdot \frac{T_c}{2}$$

- 0
- 0
- 0
- 0
- 0
- 0
- 0
- 0
- 0
- 0
- 0

•  $a_1$  can be computed

# If we integrate in [ 0 to Tc ] (3)



$$\int_0^{T_c} f(t) \times \cos(1\omega t) dt$$

0

0

$$b_1 \cdot \frac{T_c}{2}$$

0

0

0

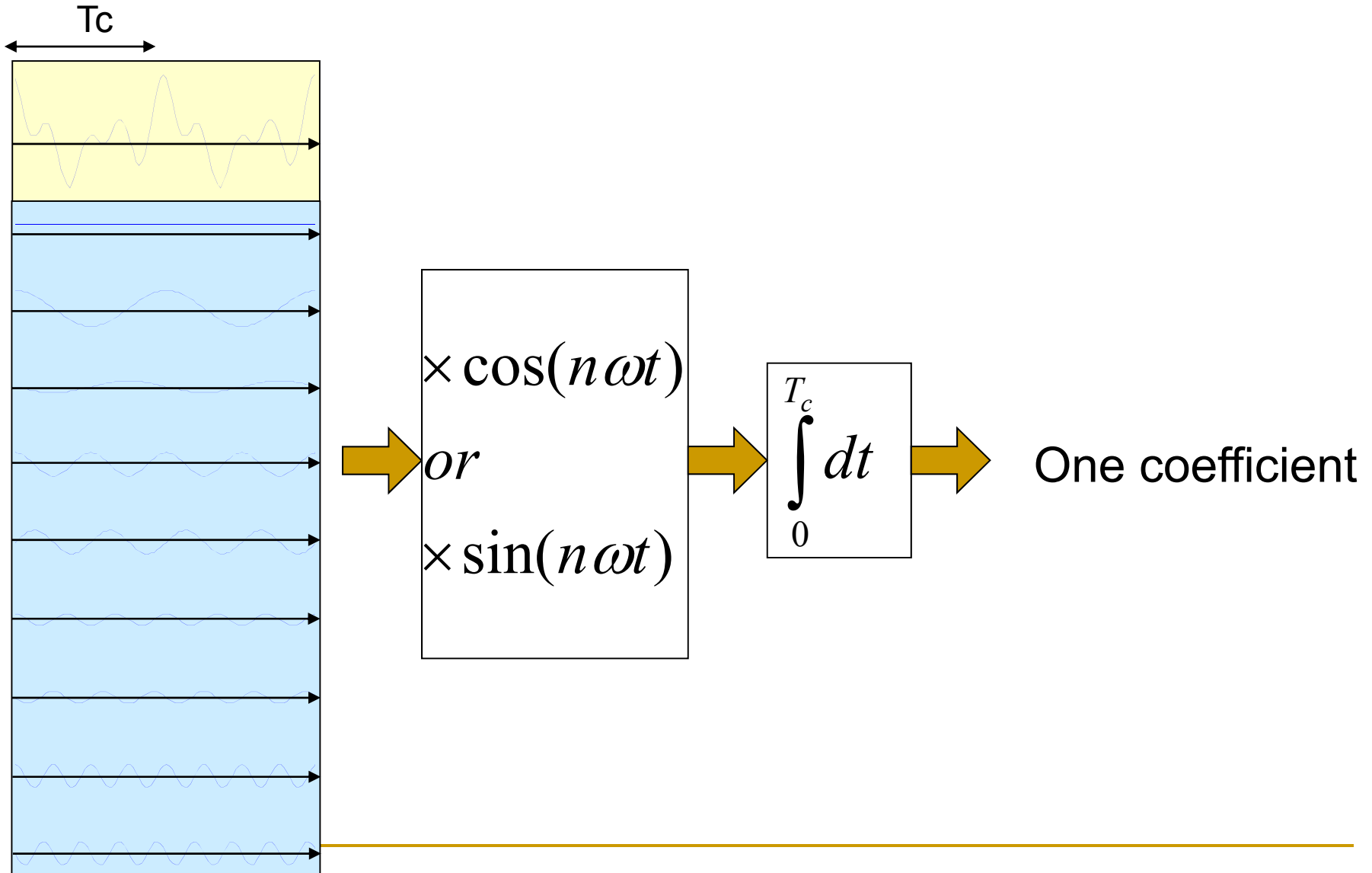
0

0

0

•  $b_1$  can be computed

# By changing multiplier, each coefficient computed



# Sinusoidal Orthogonality

- $m, n$ : integer,  $T_c = 1/f_0$

$$\int_0^{T_c} \cos(2\pi m f_0 t) \cdot \cos(2\pi n f_0 t) dt = \begin{cases} \frac{T_c}{2} & (m = n) \\ 0 & (m \neq n) \end{cases} \quad \longrightarrow \text{Orthogonal}$$

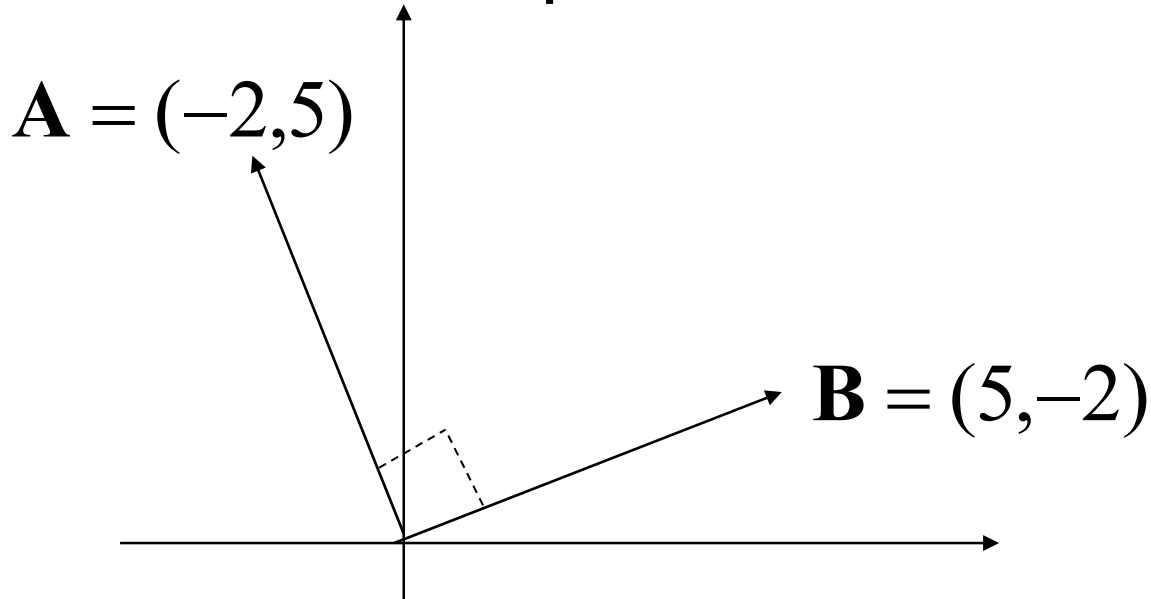
$$\int_0^{T_c} \sin(2\pi m f_0 t) \cdot \sin(2\pi n f_0 t) dt = \begin{cases} \frac{T_c}{2} & (m = n) \\ 0 & (m \neq n) \end{cases} \quad \longrightarrow \text{Orthogonal}$$

$$\int_0^{T_c} \cos(2\pi m f_0 t) \cdot \sin(2\pi n f_0 t) dt = 0 \quad \longrightarrow \text{Orthogonal}$$



# Another Orthogonality (1)

- Vector inner product



$$\mathbf{A} \cdot \mathbf{B} = (-2) \times 5 + 5 \times 2 = 0$$

→ Orthogonal

$$= |\mathbf{A}| \cdot |\mathbf{B}| \cos \theta$$

$\theta = 90$  degree

# Another Orthogonality (2)

- n dimensional vector

$$\mathbf{A} = (a_1, a_2, \dots, a_n)$$

$$\mathbf{B} = (b_1, b_2, \dots, b_n)$$

$$\mathbf{A} \cdot \mathbf{B} = a_1 \cdot b_1 + a_2 \cdot b_2 + \dots + a_n \cdot b_n$$

**IF  $\mathbf{A} \cdot \mathbf{B} = 0$**

**THEN A and B are Orthogonal.**

$\int_0^{T_c} dt$  is same as the N dim inner product

*If signal is sampled,*

$$\int_0^{T_c} \cos(2\pi m f_0 t) \cdot \cos(2\pi n f_0 t) dt$$

$$= \sum_{i=0}^{N-1} \cos(2\pi m f_0 \frac{i}{N} T_c) \cos(2\pi n f_0 \frac{i}{N} T_c)$$

$$= \sum_{i=0}^{N-1} \cos(2\pi m \frac{i}{N}) \cos(2\pi n \frac{i}{N})$$

$$= x_0 \cdot y_0 + x_1 \cdot y_1 + \dots + x_{N-1} \cdot y_{N-1}$$

Freq= $n\omega$ (Hz) sinusoids are Orthogonal each other ( $n$ =integer)

# Fourier Series Summary

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t))$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n \frac{2\pi}{T_c} t) + b_n \sin(n \frac{2\pi}{T_c} t))$$

$$a_0 = \frac{1}{T_c} \int_0^{T_c} f(t) dt$$

$$a_n = \frac{2}{T_c} \int_0^{T_c} f(t) \cos(n\omega t) dt$$

$$b_n = \frac{2}{T_c} \int_0^{T_c} f(t) \sin(n\omega t) dt$$

# Complex form Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{jn\omega t}$$

$$c_n = \frac{1}{T_c} \int_0^{T_c} f(t) e^{-jn\omega t} dt$$

$$\int_0^{T_c} e^{jn\omega t} e^{-jm\omega t} dt$$

$$= \int_0^{T_c} e^{j\omega(n-m)t} dt = \begin{cases} T_c & (m = n) \\ 0 & (m \neq n) \end{cases} \rightarrow \text{Orthogonal}$$

# HW2

$$f(t) = \begin{cases} 1 & 0 \leq t \leq \frac{1}{2}T_0 \\ -1 & \frac{1}{2}T_0 \leq t < T_0 \end{cases}$$

- [2-1] Compute the complex form Fourier Series coefficient  $c_n$  for  $f(x)$ .
- [2-2] Draw the Spectrum of  $f(t)$  when  $T_0=0.04\text{sec}$ .

2.30