Matrix Based OFDM modeling and Introduction to MIMO modeling

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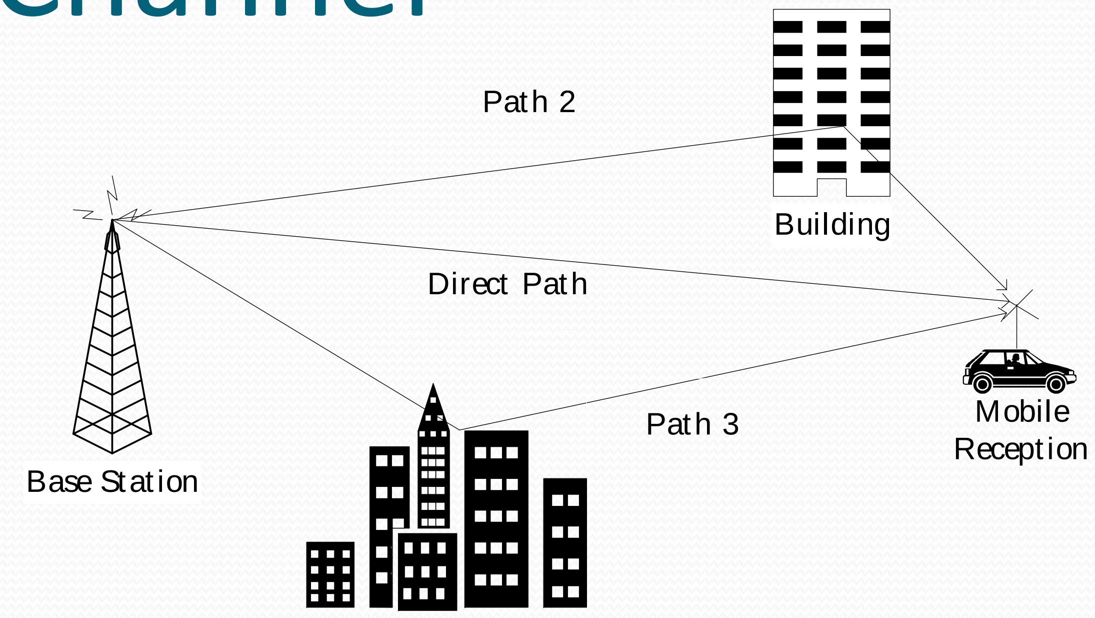
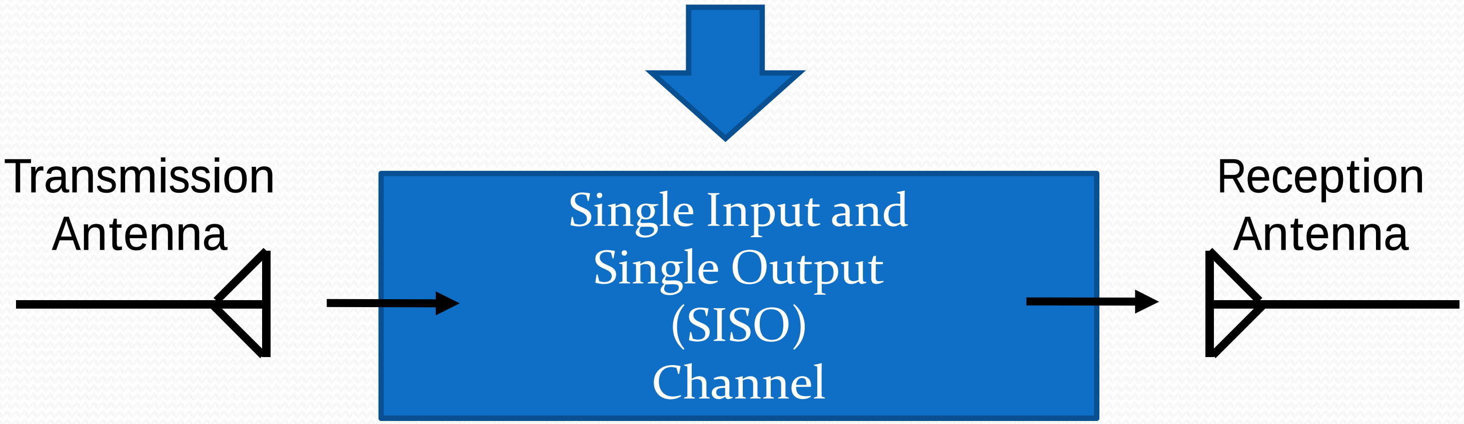
Section 1

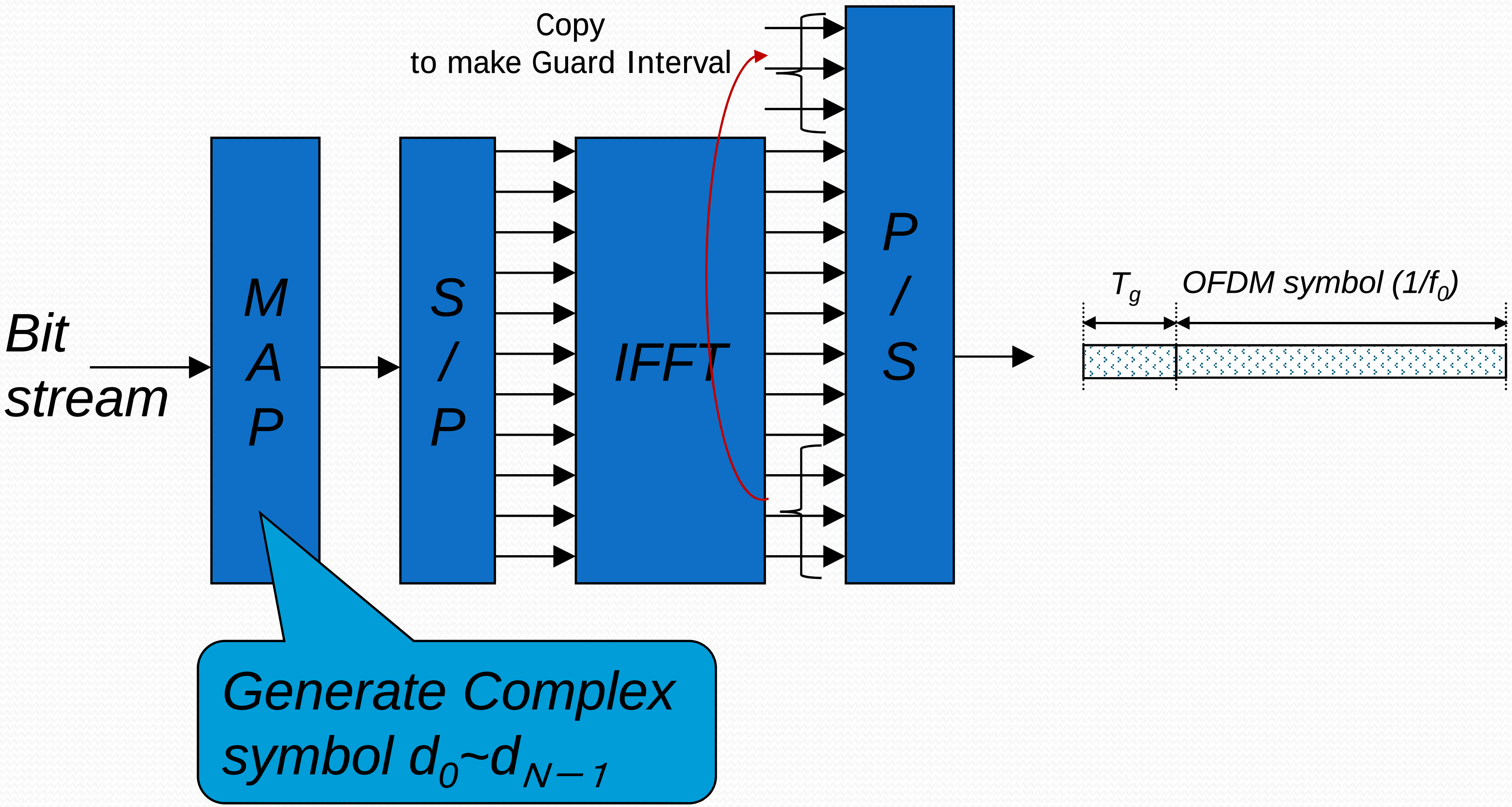
- Matrix Based OFDM Modeling
- Channel Matrix diagonalization by Unitary Matrix FFT

SISO Channel

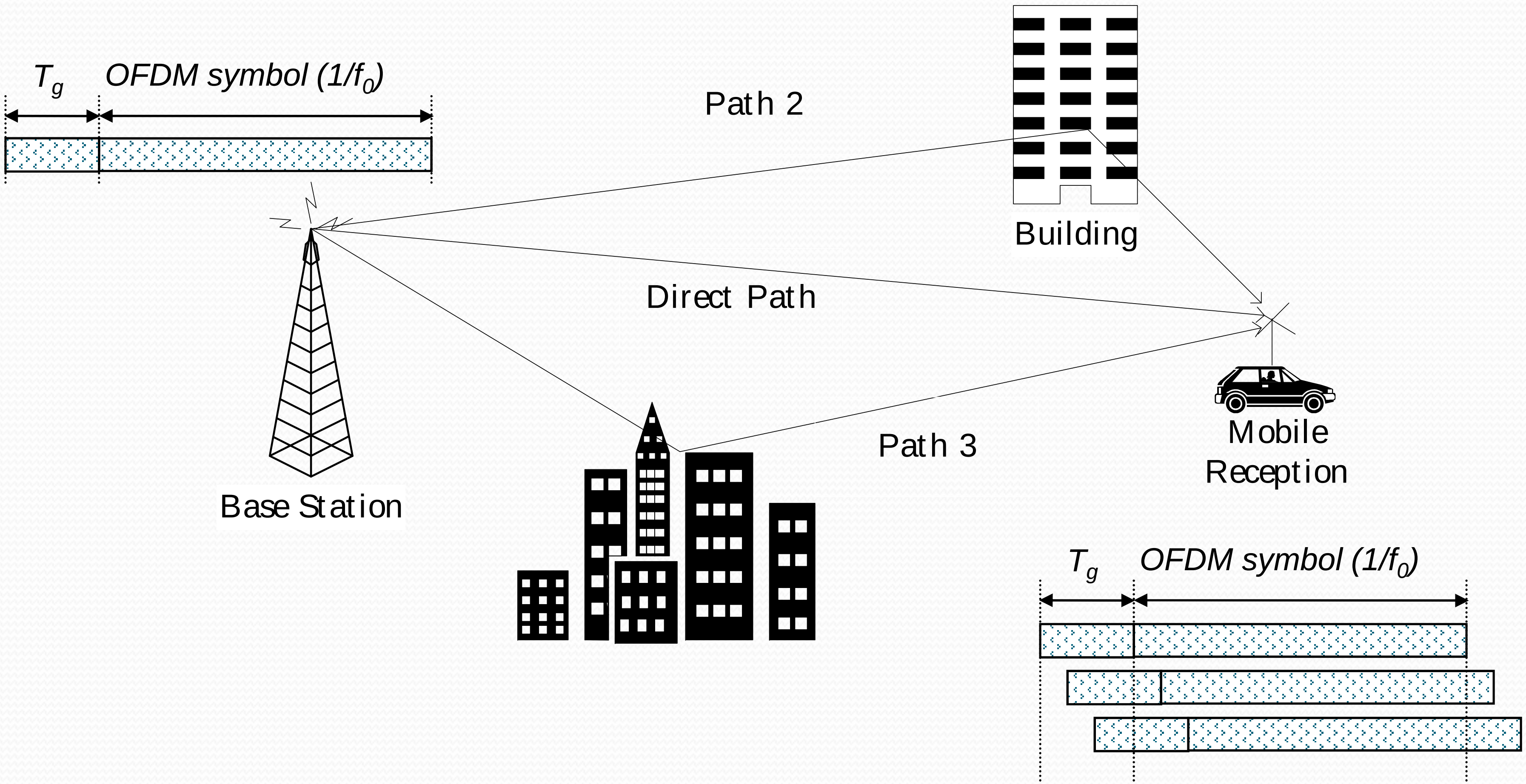




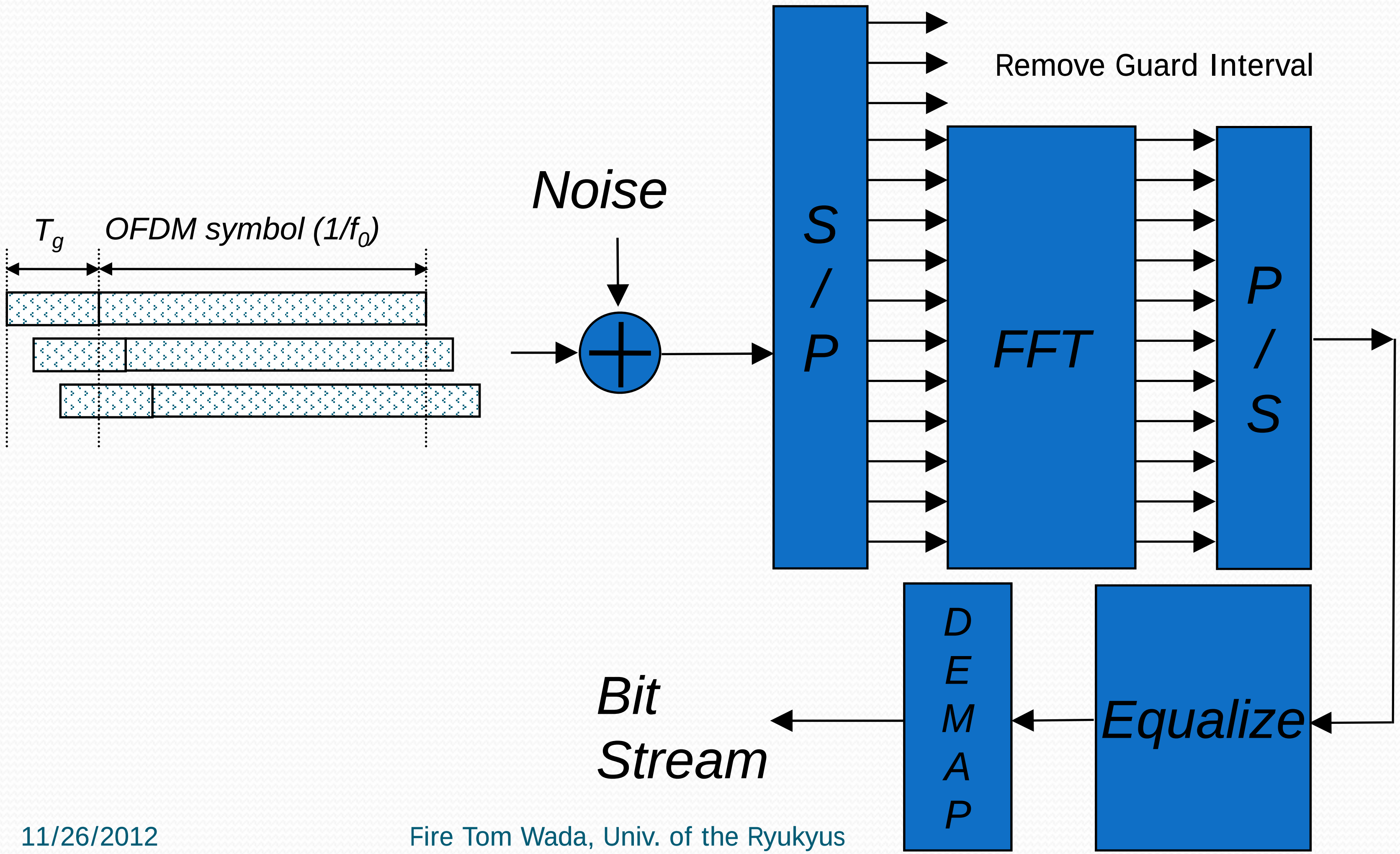
OFDM Modulator



Multi-path channel



OFDM Demodulator



FFT matrix

$$\begin{pmatrix} Y(0) \\ Y(1) \\ \vdots \\ Y(N-1) \end{pmatrix} = [FFT] \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{pmatrix} = \frac{1}{\sqrt{N}} \left[\omega^{-(k-1)*(l-1)}; k-row, l-column \right] \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{pmatrix}$$

$$\begin{pmatrix} Y(0) \\ Y(1) \\ \vdots \\ Y(N-1) \end{pmatrix} = \frac{1}{\sqrt{N}} \begin{pmatrix} \omega^0 & \omega^0 & \cdots & \omega^0 \\ \omega^0 & \omega^{-1} & \cdots & \omega^{-(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^0 & \omega^{-(N-1)} & \cdots & \omega^{-(N-1)*(N-1)} \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{pmatrix}$$

$$Here, \omega = e^{j\frac{2\pi}{N}}$$

IFFT matrix

$$\begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{pmatrix} = [IFFT] \begin{pmatrix} Y(0) \\ Y(1) \\ \vdots \\ Y(N-1) \end{pmatrix} = \frac{1}{\sqrt{N}} \left[\omega^{(k-1)*(l-1)}; k-row, l-column \right] \begin{pmatrix} Y(0) \\ Y(1) \\ \vdots \\ Y(N-1) \end{pmatrix}$$

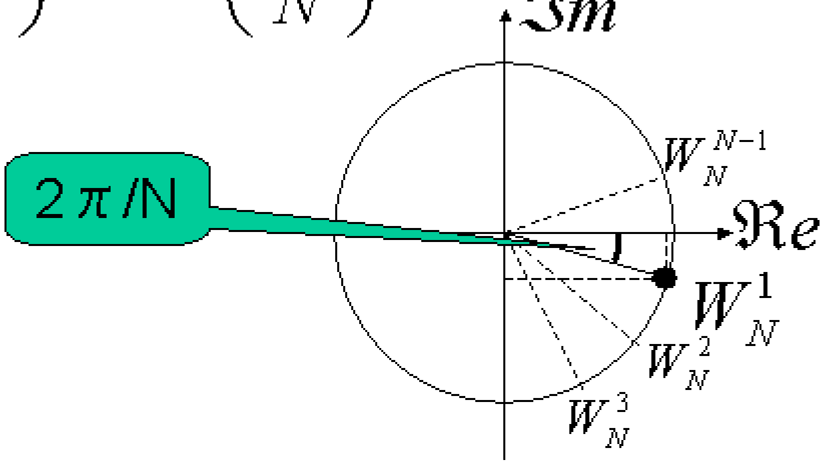
$$\begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{pmatrix} = \frac{1}{\sqrt{N}} \begin{pmatrix} \omega^0 & \omega^0 & \cdots & \omega^0 \\ \omega^0 & \omega^1 & \cdots & \omega^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^0 & \omega^{(N-1)} & \cdots & \omega^{(N-1)*(N-1)} \end{pmatrix} \begin{pmatrix} Y(0) \\ Y(1) \\ \vdots \\ Y(N-1) \end{pmatrix}$$

$$Here, \omega = e^{j\frac{2\pi}{M}}$$

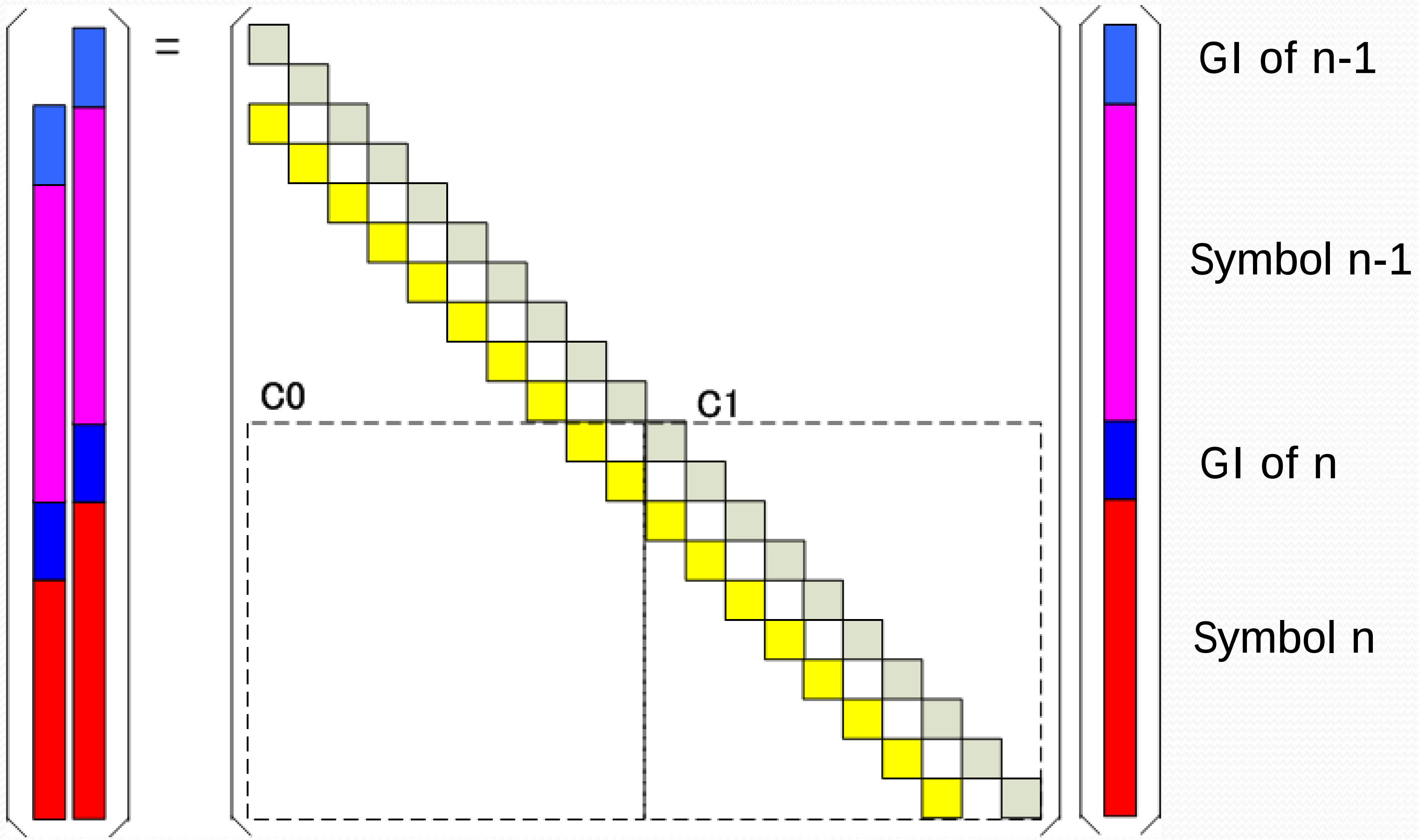
Twiddle Factor W_N^{nk}

$$W_N = e^{-j\left(\frac{2\pi}{N}\right)}$$

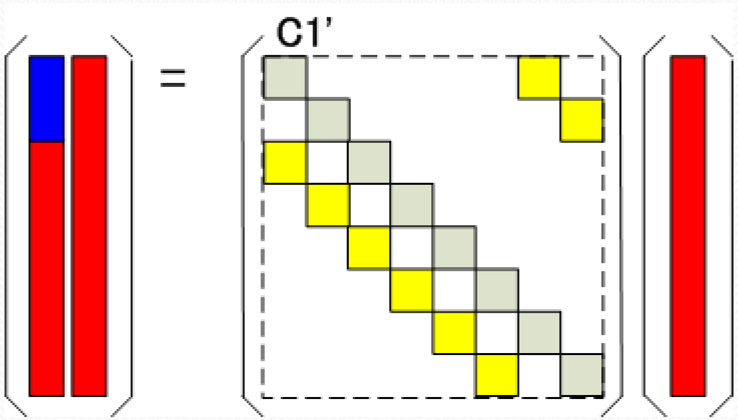
$$= \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right)$$



Multi-path channel in Matrix

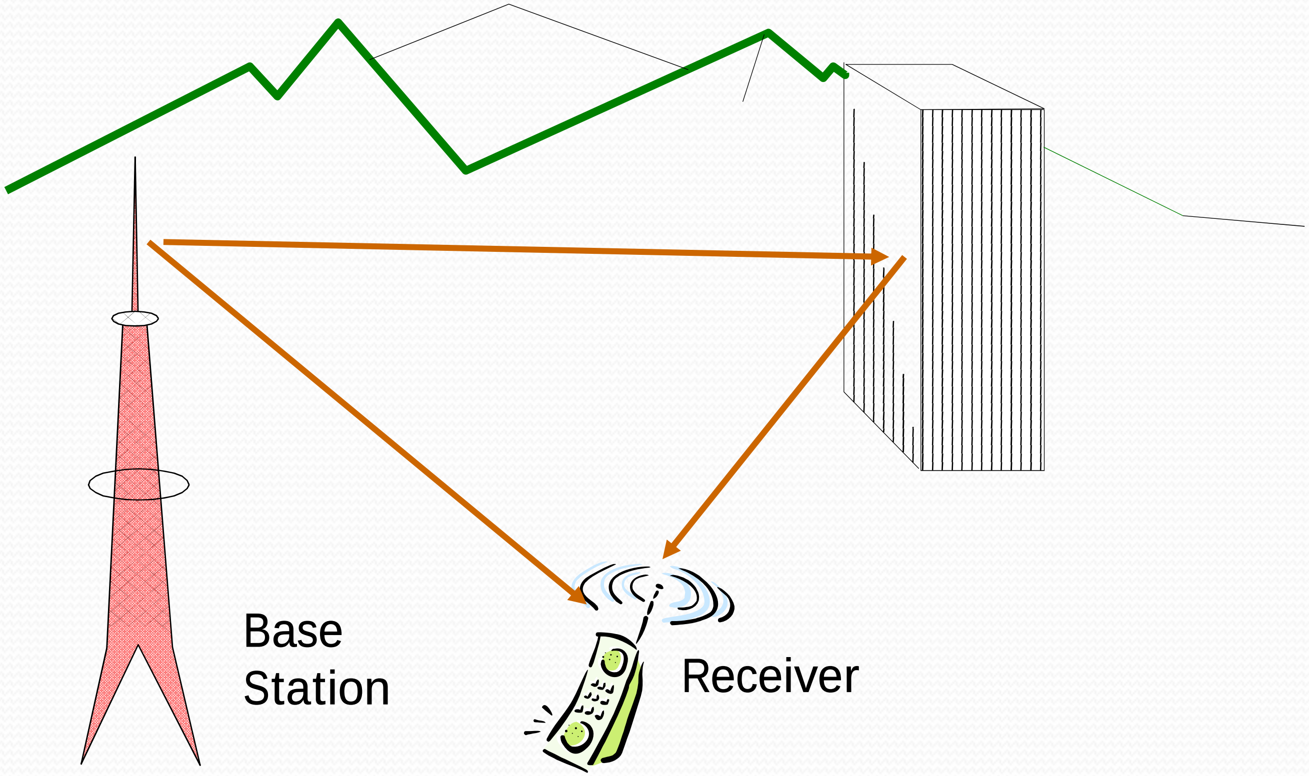


If Multi-path delay is small than GI length



- Channel Matrix is Cyclic Matrix by GI.

Two path Multi path Channel Example



Channel Impulse Response = [1, 0.5 , 0, 0]

Two path Multi path Channel Example

$$\begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix} = FFT * Channel * IFFT * \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}$$

$$\begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix} = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0.5 \\ 0.5 & 1 & 0 & 0 \\ 0 & 0.5 & 1 & 0 \\ 0 & 0 & 0.5 & 1 \end{pmatrix} \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{9} \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}$$

$$\begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix} = \begin{pmatrix} H(0) & 0 & 0 & 0 \\ 0 & H(1) & 0 & 0 \\ 0 & 0 & H(2) & 0 \\ 0 & 0 & 0 & H(3) \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}$$

If time domain channel matrix is cyclic, Frequency Domain Channel Matrix is diagonal!

Additive Noise

$$\begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix} = FFT * \left[Channel * IFFT * \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix} + \begin{pmatrix} noise(0) \\ noise(1) \\ noise(2) \\ noise(3) \end{pmatrix} \right]$$

$$\begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix} = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0.5 \\ 0.5 & 1 & 0 & 0 \\ 0 & 0.5 & 1 & 0 \\ 0 & 0 & 0.5 & 1 \end{pmatrix} \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{9} \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix} + \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} \end{pmatrix} \begin{pmatrix} noise(0) \\ noise(1) \\ noise(2) \\ noise(3) \end{pmatrix}$$

$$\begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix} = \begin{pmatrix} H(0) & 0 & 0 & 0 \\ 0 & H(1) & 0 & 0 \\ 0 & 0 & H(2) & 0 \\ 0 & 0 & 0 & H(3) \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix} + \begin{pmatrix} N(0) \\ N(1) \\ N(2) \\ N(3) \end{pmatrix}$$

How to recover sending signal from receiver signal. - EQUALIZE -

$$\mathit{Ignore\quad Noise}$$

$$\begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix} = FFT * Channel * IFFT * \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix} = \begin{pmatrix} H(0) & 0 & 0 & 0 \\ 0 & H(1) & 0 & 0 \\ 0 & 0 & H(2) & 0 \\ 0 & 0 & 0 & H(3) \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}$$

$$\mathit{Then}$$

$$\begin{pmatrix} \hat{X}(0) \\ \hat{X}(1) \\ \hat{X}(2) \\ \hat{X}(3) \end{pmatrix} = \begin{pmatrix} \frac{1}{H(0)} & 0 & 0 & 0 \\ 0 & \frac{1}{H(1)} & 0 & 0 \\ 0 & 0 & \frac{1}{H(2)} & 0 \\ 0 & 0 & 0 & \frac{1}{H(3)} \end{pmatrix} \begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix}$$

Summary of Matrix model of OFDM

$$\begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix} \Rightarrow \left(\quad IFFT \quad \right) \Rightarrow \left(\quad \begin{matrix} Cyclic \\ Channel \end{matrix} \quad \right) \Rightarrow \left(\quad FFT \quad \right) \Rightarrow \begin{pmatrix} \frac{1}{H(0)} & 0 & 0 & 0 \\ 0 & \frac{1}{H(1)} & 0 & 0 \\ 0 & 0 & \frac{1}{H(2)} & 0 \\ 0 & 0 & 0 & \frac{1}{H(3)} \end{pmatrix} \Rightarrow \begin{pmatrix} \hat{X}(0) \\ \hat{X}(1) \\ \hat{X}(2) \\ \hat{X}(3) \end{pmatrix}$$

$$\begin{pmatrix} \hat{X}(0) \\ \hat{X}(1) \\ \hat{X}(2) \\ \hat{X}(3) \end{pmatrix} = \begin{pmatrix} \frac{1}{H(0)} & 0 & 0 & 0 \\ 0 & \frac{1}{H(1)} & 0 & 0 \\ 0 & 0 & \frac{1}{H(2)} & 0 \\ 0 & 0 & 0 & \frac{1}{H(3)} \end{pmatrix} \left[\quad FFT \quad \right] \left[\quad \begin{matrix} Cyclic \\ Channel \end{matrix} \quad \right] \left[\quad IFFT \quad \right] \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}$$

Important Mathematics

- Cyclic Matrix can be diagonalized by FFT and IFFT.
  - X^H is *Hermitian of X, that is, complex conjugate and transpose.*

$$\begin{pmatrix} \\ FFT \\ \\ \end{pmatrix}\begin{pmatrix} Cyclic \\ Matrix \end{pmatrix}\begin{pmatrix} \\ IFFT \\ \\ \end{pmatrix} = \begin{pmatrix} H(0) & 0 & 0 & 0 \\ 0 & H(1) & 0 & 0 \\ 0 & 0 & H(2) & 0 \\ 0 & 0 & 0 & H(3) \end{pmatrix} = diag(H(0), H(1), \cdots)$$

Since

$$\begin{pmatrix} \\ FFT \\ \\ \end{pmatrix}^H = \begin{pmatrix} \\ IFFT \\ \\ \end{pmatrix} = U$$

$$U\begin{pmatrix} Cyclic \\ Matrix \end{pmatrix}U^H = \begin{pmatrix} H(0) & 0 & 0 & 0 \\ 0 & H(1) & 0 & 0 \\ 0 & 0 & H(2) & 0 \\ 0 & 0 & 0 & H(3) \end{pmatrix} = diag(H(0), H(1), \cdots)$$

Unitary Matrix

- Unitary Matrix U can satisfy following property.

$$\mathbf{U}^H\mathbf{U} = \mathbf{U}\mathbf{U}^H = \mathbf{I}$$

$$when \quad \mathbf{e}_i \quad is \quad vertical \quad vector,$$

$$\mathbf{U} = [\mathbf{e}_1 \quad \cdots \quad \mathbf{e}_N]$$

$$then$$

$$\mathbf{e}_i{}^H\mathbf{e}_j = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

- Eigen value of Channel Cyclic Matrix is Channel Transfer Function as (H(0), H(1), H(2), …).

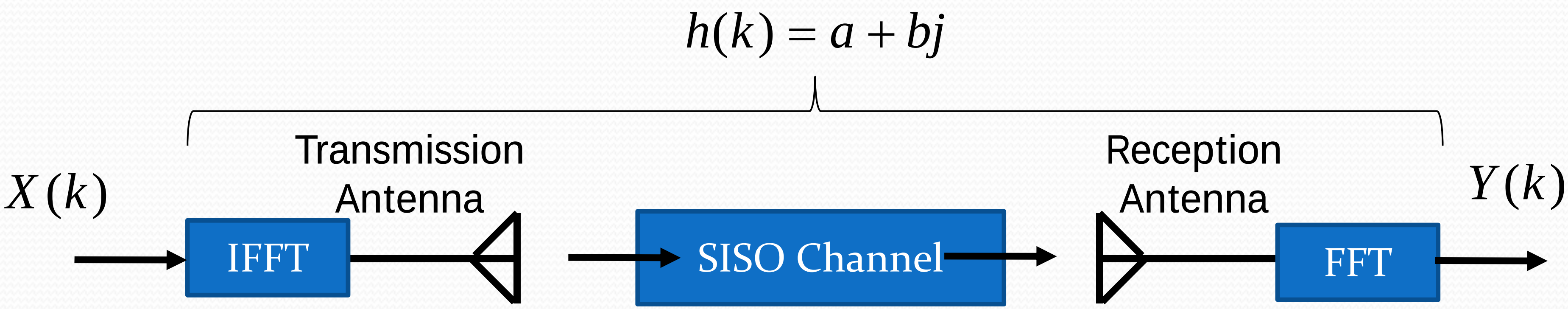
$$\begin{pmatrix} Cyclic \\ Matrix \end{pmatrix} = U^H \begin{pmatrix} H(0) & 0 & 0 & 0 \\ 0 & H(1) & 0 & 0 \\ 0 & 0 & H(2) & 0 \\ 0 & 0 & 0 & H(3) \end{pmatrix} U$$

Section 2

- MIMO Channel Modeling

SISO Channel

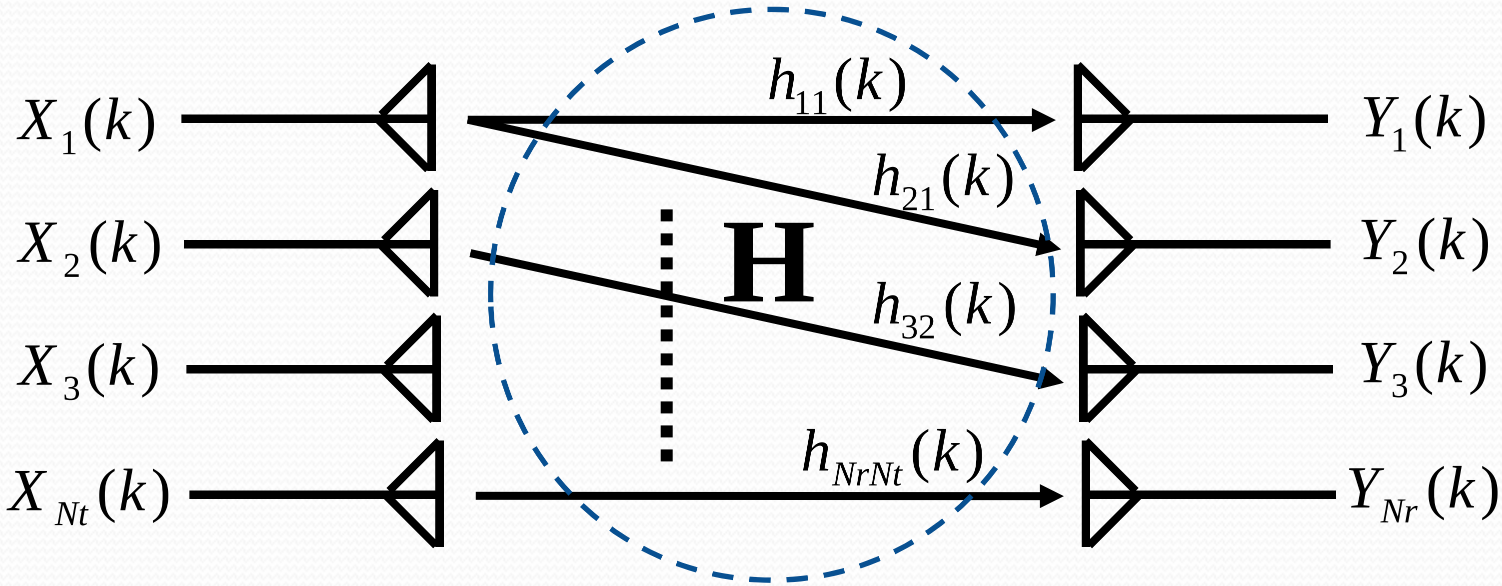
- OFDM makes Multi-path channel simple complex h(k) for freq=k.



$$\begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix} = FFT * Channel * IFFT * \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix} = \begin{pmatrix} h(0) & 0 & 0 & 0 \\ 0 & h(1) & 0 & 0 \\ 0 & 0 & h(2) & 0 \\ 0 & 0 & 0 & h(3) \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}$$

$$Y(k) = h(k)X(k)$$

MIMO Channel
- Nr X Nt SISO Channels for Freq=k -



$$\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{Nr} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & 0 & h_{1Nt} \\ h_{21} & h_{22} & 0 & h_{2Nt} \\ 0 & 0 & & \\ h_{Nr1} & h_{Nr2} & 0 & h_{NrNt} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{Nt} \end{pmatrix} = \mathbf{H} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{Nt} \end{pmatrix}$$

Singular value decomposition of Nr x Nt Matrix **H**

- Nr x Nt matrix **H** can be decomposed as below using Nr x Nr Unitary matrix **V** and Nt x Nt Unitary matrix **U**.
- Σ is Nr x Nt diagonal matrix.

$$\mathbf{H} = \mathbf{V}\Sigma\mathbf{U}^H$$

$$= \begin{bmatrix} & & \\ & Nr \times Nr & \\ & & \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_K} & 0 & 0 \end{bmatrix} \begin{bmatrix} & & \\ & Nt \times Nt & \\ & & \end{bmatrix}$$

$$\lambda_1, \cdots \lambda_K \ \textit{is eigen value of} \ \mathbf{H}^H\mathbf{H} \ \textit{and} \ \mathbf{H}\mathbf{H}^H .$$

SVD Example by Matlab(1)

```
H =

     1     2     3
     2     4     5

>> [U,S,V] = svd(H)
U =
   -0.4863   -0.8738
   -0.8738    0.4863
S =

    7.6756         0         0
         0    0.2913         0
V =
   -0.2910    0.3396   -0.8944
   -0.5821    0.6791    0.4472
   -0.7593   -0.6508   -0.0000
```

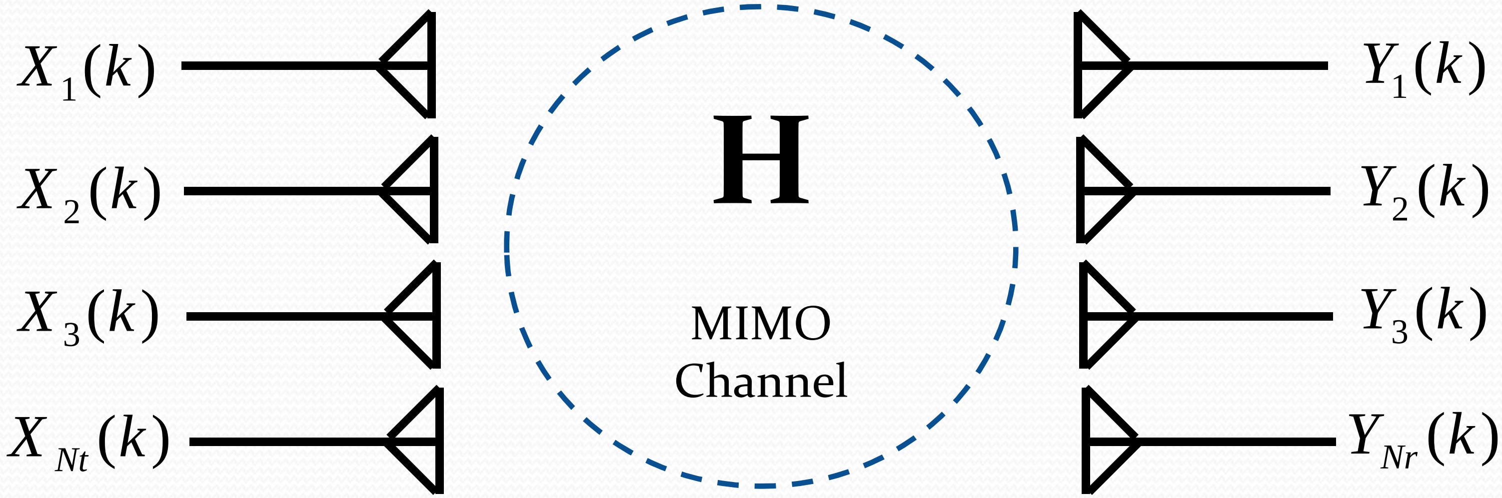
```
H =
     1     2
     2     4
     3     5

>> [U,S,V] = svd(H)
U =

   -0.2910   -0.3396   -0.8944
   -0.5821   -0.6791    0.4472
   -0.7593    0.6508   -0.0000
S =

    7.6756         0
         0    0.2913
         0         0
V =

   -0.4863    0.8738
   -0.8738   -0.4863
```

SVD Example by Matlab(2)

```
H =
    1.0000 + 1.0000i   2.0000 + 1.0000i
    1.0000 - 3.0000i   3.0000 - 1.0000i

>> [U,S,V] = svd(H)
U =
  -0.4616 - 0.0659i  -0.4907 + 0.7361i
  -0.3956 + 0.7913i  -0.2863 - 0.3680i
S =
     5.0000         0
          0    1.4142
V =
  -0.6594             0.7518
  -0.5934 + 0.4616i  -0.5205 + 0.4048i

>> U*S*V'

ans =

    1.0000 + 1.0000i   2.0000 + 1.0000i
    1.0000 - 3.0000i   3.0000 - 1.0000i
```

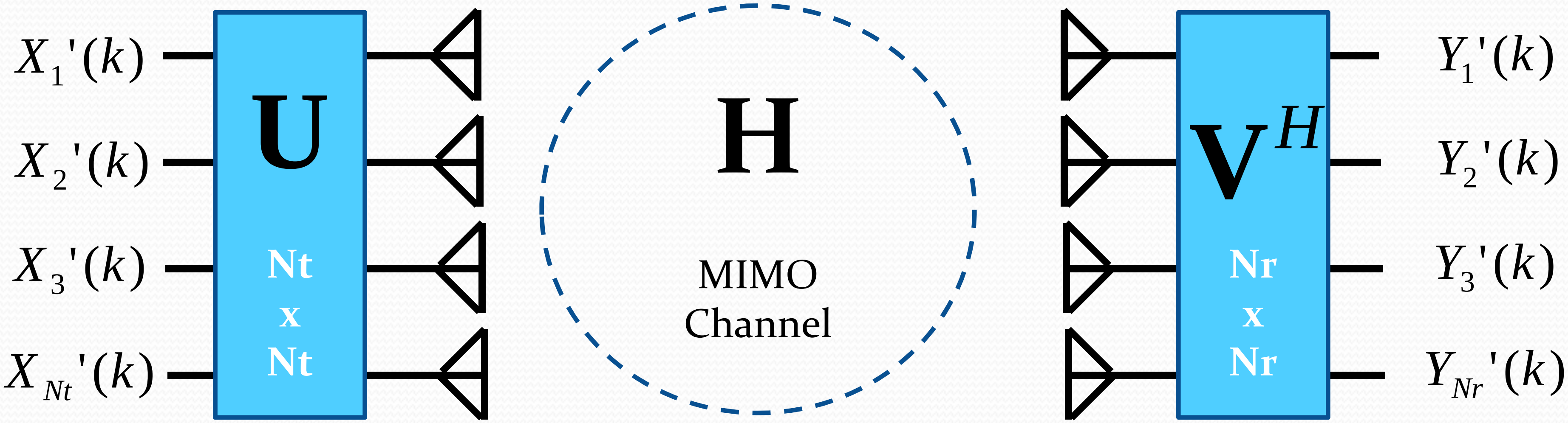
```
>> U*S*V'

ans =

    1.0000 + 1.0000i   2.0000 + 1.0000i
    1.0000 - 3.0000i   3.0000 - 1.0000i

>> U'*U
ans =

    1.0000             0.0000 - 0.0000i
    0.0000 + 0.0000i   1.0000

>> U*U'
ans =

    1.0000             0.0000 - 0.0000i
    0.0000 + 0.0000i   1.0000
```

MIMO communication



$$\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{Nr} \end{pmatrix} = \mathbf{H} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{Nt} \end{pmatrix}$$

Introduce pre-processing and post-processing



$$\begin{pmatrix} Y_1' \\ Y_2' \\ \vdots \\ Y_{Nr}' \end{pmatrix} = \mathbf{V}^H \mathbf{H} \mathbf{U} \begin{pmatrix} X_1' \\ X_2' \\ \vdots \\ X_{Nt}' \end{pmatrix}$$

There are K(=rank(H)) independent channel

$$\begin{pmatrix} Y_1' \\ Y_2' \\ \vdots \\ Y_{Nr}' \end{pmatrix} = \mathbf{V}^H \mathbf{H} \mathbf{U} \begin{pmatrix} X_1' \\ X_2' \\ \vdots \\ X_{Nt}' \end{pmatrix}$$

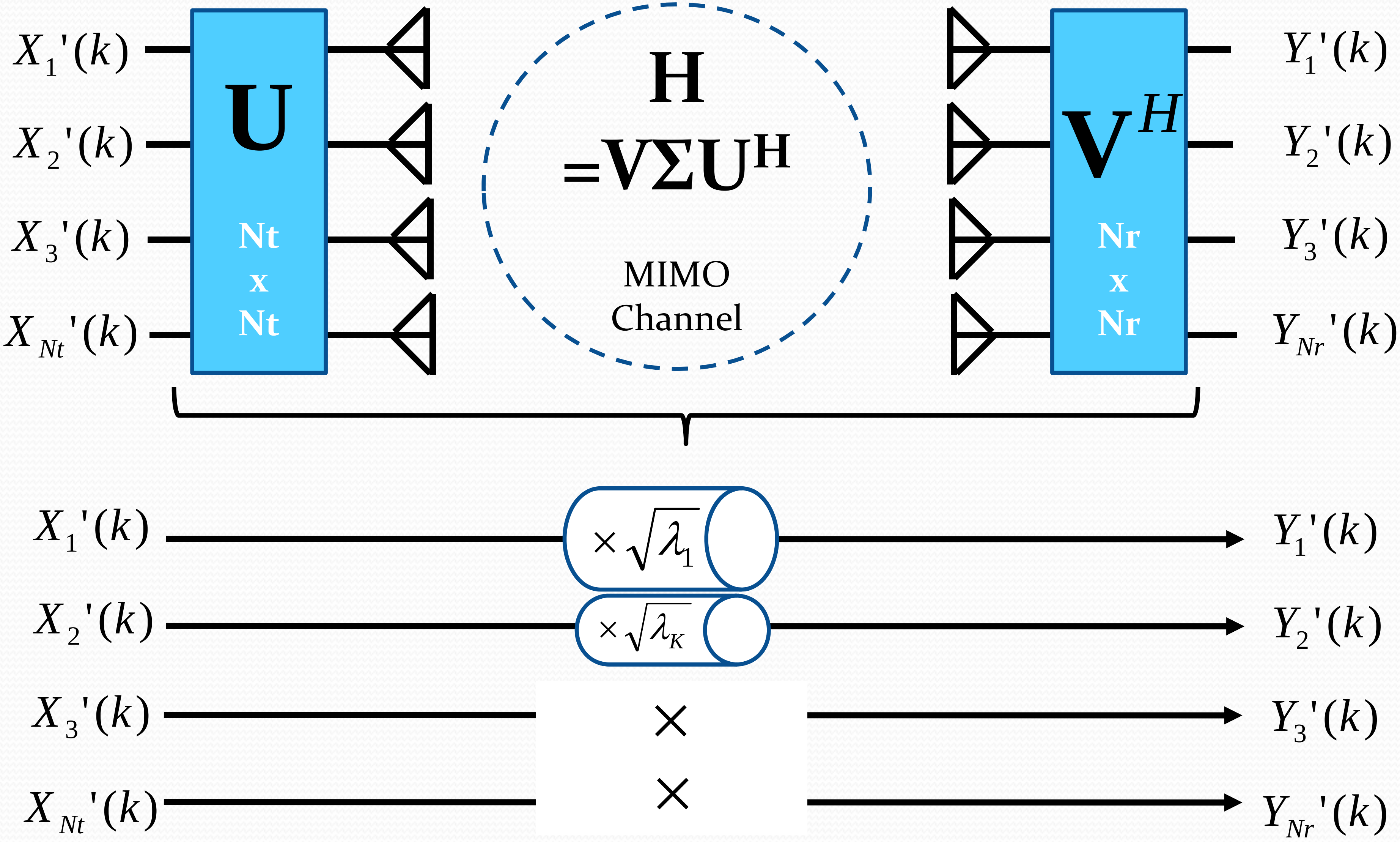
$$\begin{pmatrix} Y_1' \\ Y_2' \\ \vdots \\ Y_{Nr}' \end{pmatrix} = \mathbf{V}^H \mathbf{V} \Sigma \mathbf{U}^H \mathbf{U} \begin{pmatrix} X_1' \\ X_2' \\ \vdots \\ X_{Nt}' \end{pmatrix}$$

$$\begin{pmatrix} Y_1' \\ Y_2' \\ \vdots \\ Y_{Nr}' \end{pmatrix} = \Sigma \begin{pmatrix} X_1' \\ X_2' \\ \vdots \\ X_{Nt}' \end{pmatrix}$$

$$\begin{pmatrix} Y_1' \\ Y_2' \\ \vdots \\ Y_{Nr}' \end{pmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sqrt{\lambda_K} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} X_1' \\ X_2' \\ \vdots \\ X_{Nt}' \end{pmatrix}$$

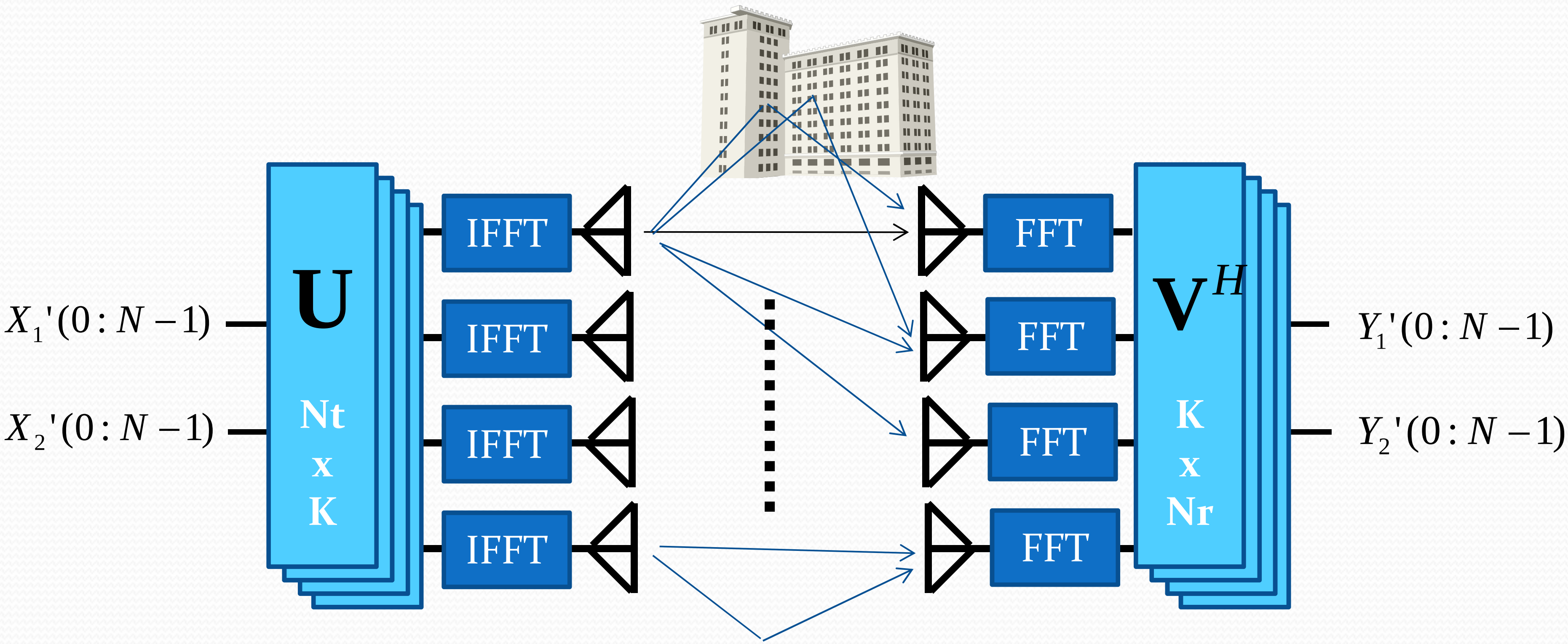
$$\begin{gathered} Y_1' = \sqrt{\lambda_1} \times X_1' \\ \vdots \\ Y_K' = \sqrt{\lambda_K} \times X_K' \\ Y_{K+1}' = 0 \\ \vdots \\ Y_{Nr}' = 0 \end{gathered}$$

SVD-MIMO system



Put them altogether MIMO-OFDM system

- Space Division Multiplexing by MIMO (K stream)
- Orthogonal Frequency Division Multplxing (OFDM)



Summary

- This presentation shows matrix based modeling for both OFDM and MIMO and there are many similarity in mathematics.
  1. OFDM realizes many parallel communication channels in frequency domain.
  2. OFDM converts multi-path channel to simple one tap channel such as h(k)=a+bj for Frequency=k.
  3. Then OFDM-based MIMO system can focus on simple channel matrix.
  4. By singular value decomposition (SVD), MIMO channel matrix H can be decomposed to $V^*\Sigma^*U^H$.
  5. Non-zero elements of Σ (rank of H) indicates parallel communication channel in space.