Error Correction Code (1)

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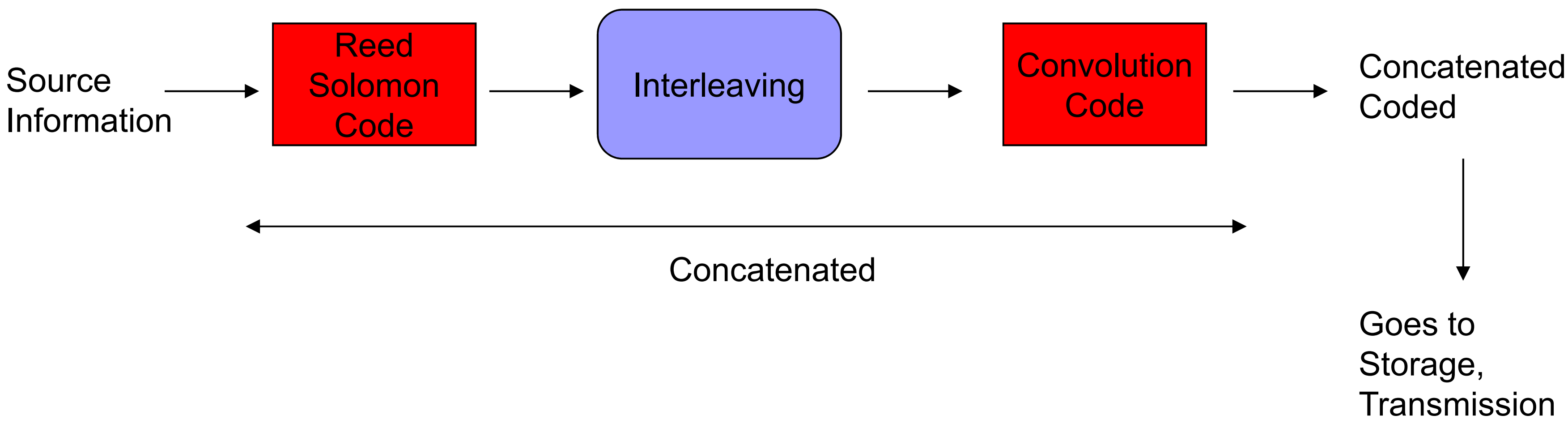
Introduction

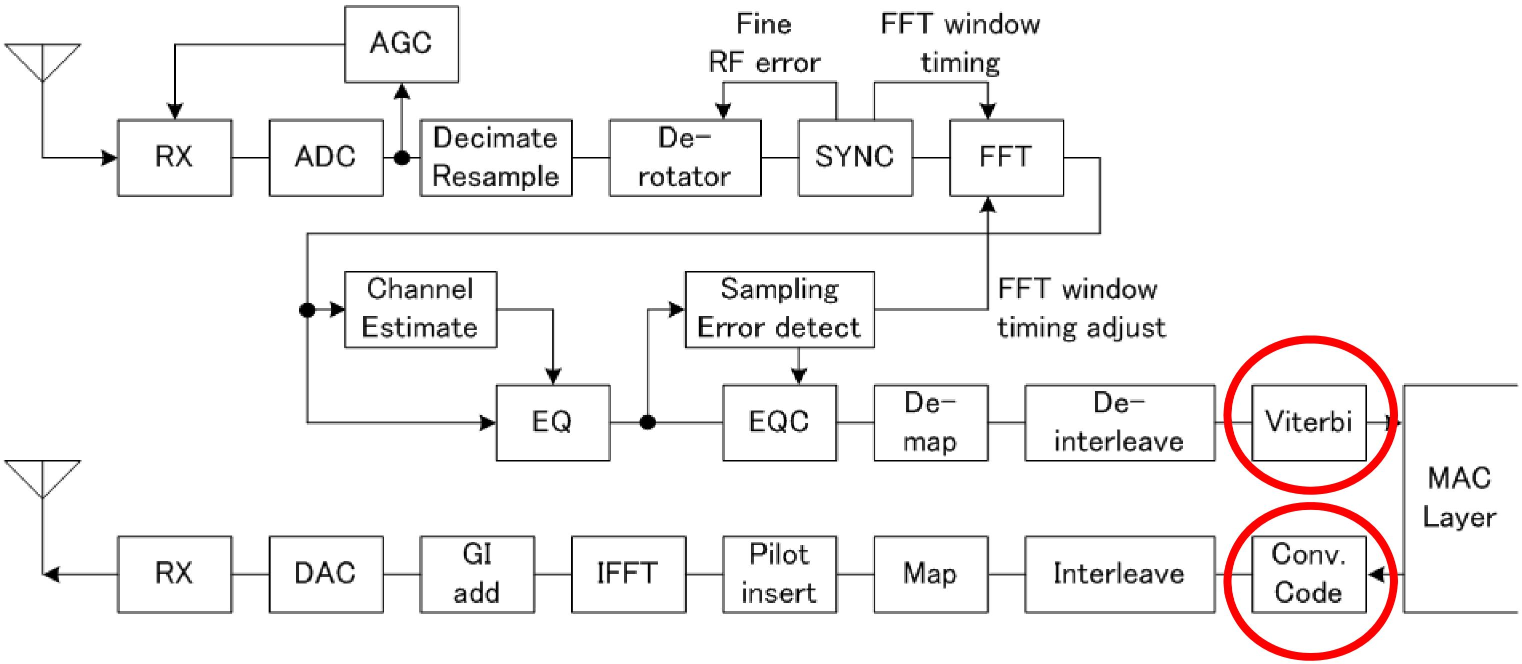
- Digital data storage
- Digital data transmission
  - Data might change by some Noise, Fading, etc.
  - Such data change have to be corrected!
- Forward-Error-Correction (FEC) is need.
  - Digital Video (DVD), Compact Disc (CD)
  - Digital communication
    - Digital Phone
    - Wireless LAN
    - Digital Broadcasting

Two major FEC technologies

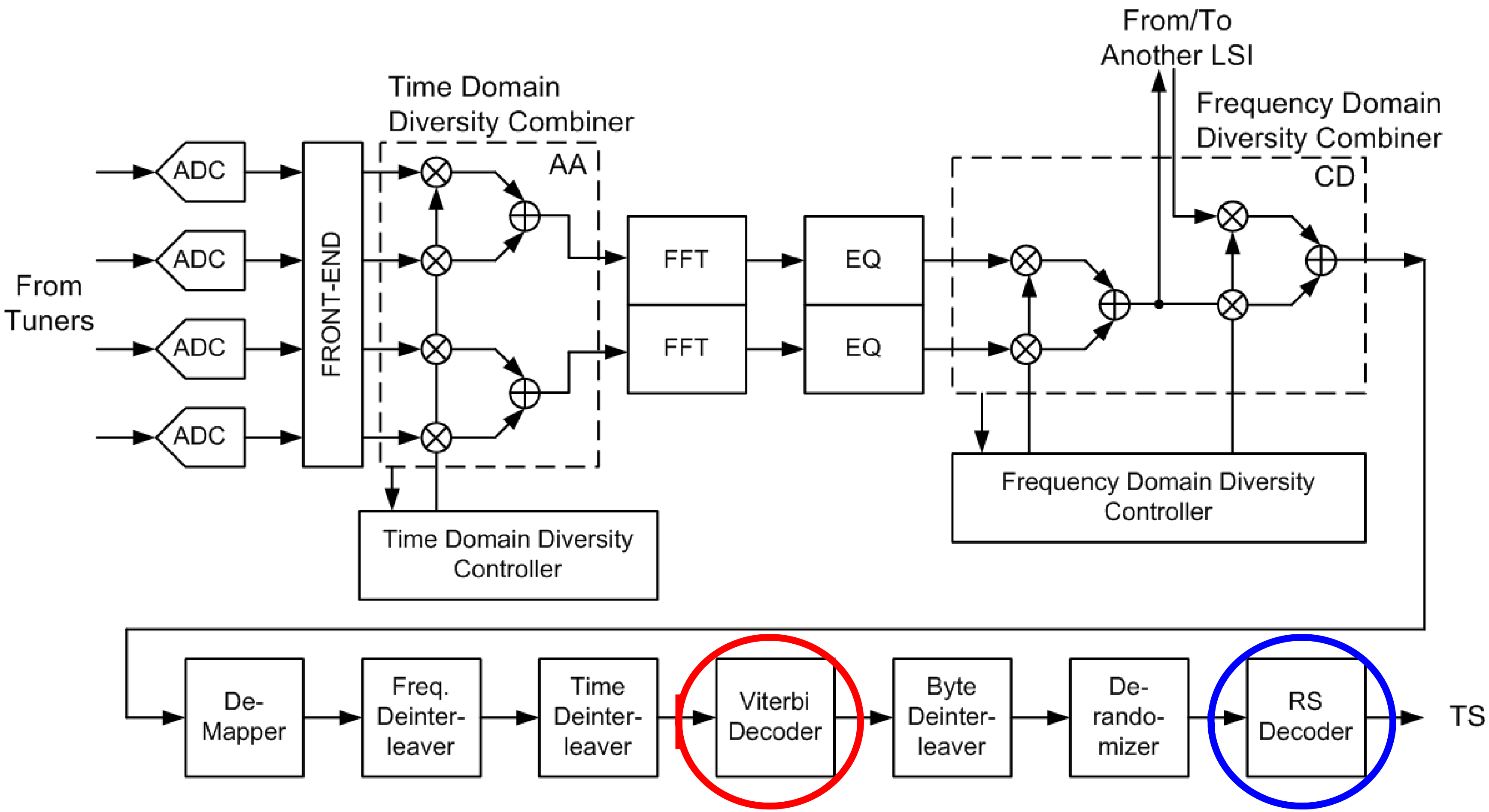
1. Reed Solomon code
2. Convolution code
3. Serially concatenated code



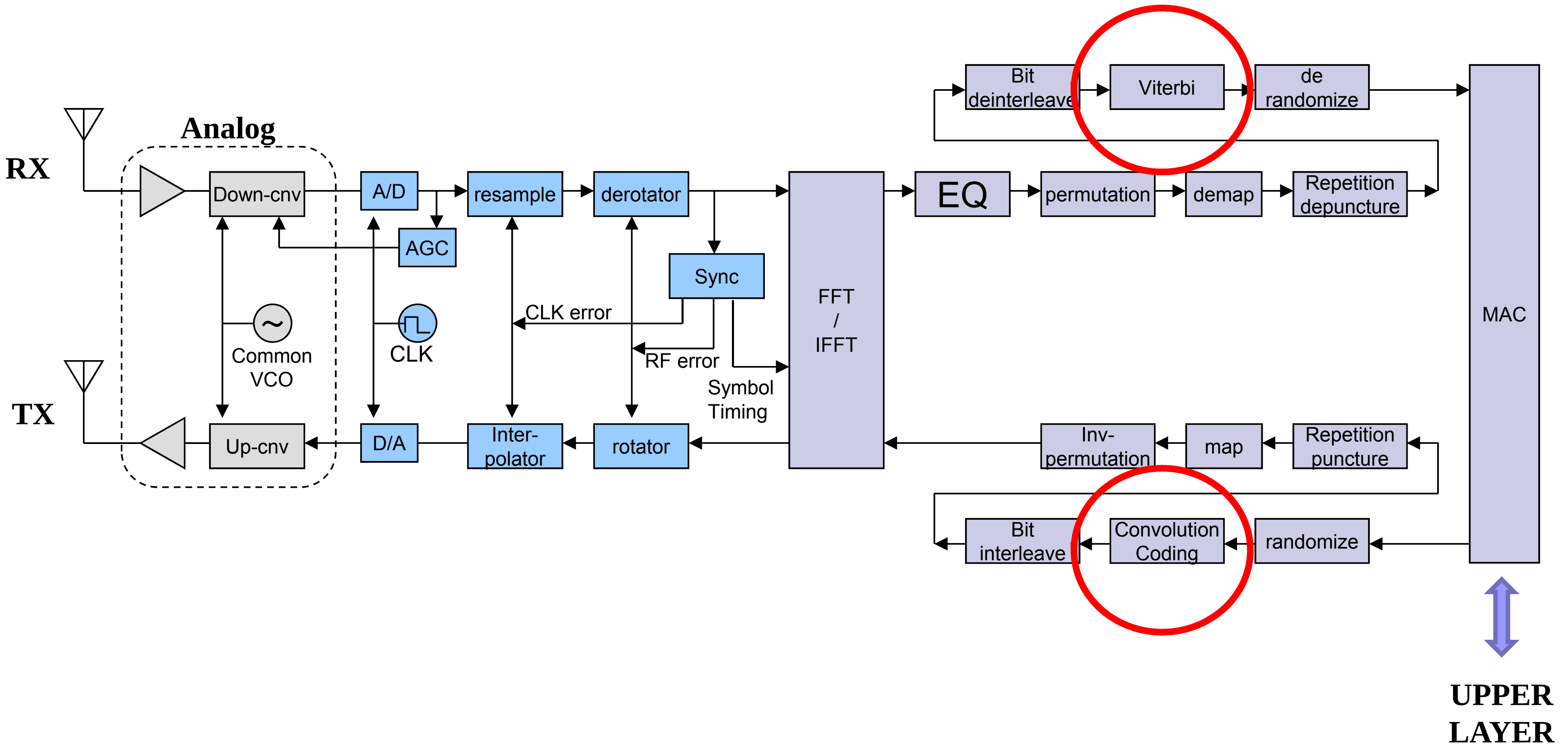
WLAN Block Diagram



18.5.2 A 2/4/8 Antennas Configurable Diversity OFDM Receiver LSI



Mobile WiMAX MS example



Reed Solomon Code

- Can correct Burst Error.
- Famous application is Compact Disc.
- Code theory based on Galois Field
  - For 8 bit = Byte information
  - Galois Field of 2^8 is used
- We will start from Galois Field in following slide.

Galois Field

- "Field" is the set in which +, -, x, / operations are possible.
  - e.g. Real numbers is Field.
  - -2.1, 0, 3, 4.5, 6, 3/2, ….
  - Number of Element is infinite（∞）.
- Instead, 8 bit digital signal can represents 2^8=256 elements only.
- Galois Field is
  - Number of element is finite. e.g. 2^8=256.
  - +, -, x, / operations are possible.
  - GF(q) means q elements Galois Fields.
  - q must be a prime number (p) or p^n .

Example 1. GF(q=2)

- GF(2)
  - Only two elements {0,1}
  - Add and Multiply : do operation and mod 2
  - Subtract : for all a = {0,1}, -a exists.
  - Division : for all a excluding {0}, a^{-1} exists.

+	0	1
0	0	1
1	1	0

Same as XOR operation

a	-a
0	0
1	1

- operation is same as +

X	0	1
0	0	0
1	0	1

Same as AND operation

a	a^{-1}
0	-
1	1

Example 2. GF(5)

- Elements = {0,1,2,3,4}
- Use mod 5 operation

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

a	-a
0	0
1	4
2	3
3	2
4	1

X	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

a	a^{-1}
0	-
1	1
2	3
3	2
4	4

- So far q=2, 5 are prime numbers.

Example 3. GF(4)

- Here 4 is NOT a prime number.
- Use mod 4
- 2^{-1} does NOT exists.

X	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

a	a^{-1}
0	-
1	1
2	-
3	3

- GF(4) with integer elements does NOT exists!
- The elements can be polynomial.

GF with polynomial elements

- Polynomial such as aX^2+bX^1+c
- Those coefficient a, b, c =GF(2)={0,1}
  - Can be added
  - Can be subtracted
  - Can be multiplied
  - Can be divided
  - And Can be modulo by other polynomial
- GF with polynomial elements is possible
- GF(4)={0X+0, 0X+1, X+0, X+1}
  - Use mod(X^2+X+1)

Example 4. $GF(2^2)$ with polynomial (1)

- Elements={0,1, x, x+1}
- Use Modulo(x^2+x+1)
- Each coefficient is GF(2)

+	0	1	x	x+1
0	0	1	x	x+1
1	1	2=0	x+1	x+2=x
x	x	x+1	2x=0	2x+1=1
x+1	x+1	x+2=x	2x+1=1	2x+2=0

a	-a
0	-
1	1
x	x
x+1	x+1

Example 4. $GF(2^2)$ with polynomial (2)

- Use Modulo(x^2+x+1)
- mod(x^2+x+1) is equivalent to use $x^2=x+1$ assignment

X	0	1	x	x+1
0	0	0	0	0
1	0	1	x	x+1
x	0	x	$x^2=x+1$	$x^2+x=$ $2x+1=1$
x+1	0	x+1	$x^2+x=$ $2x+1=1$	$x^2+2x+1=$ $x^2+1=$ $x+2=x$

a	a^{-1}
0	0
1	1
x	x+1
x+1	x

Let's calculate $(x^2+x) \bmod (x^2+x+1)$

$$
\begin{array}{r|l}
 & 1 \\
\hline
x^2+x+1 & x^2+x+0 \\
 & x^2+x+1 \\
\cline{2-2}
 & 1
\end{array}
$$

- $(x^2+x+1) \bmod (x^2+x+1)=0$
- Then $x^2+x+1=0$
- Now consider the root of $x^2+x+1=0$ is α
- Then $\alpha^2+\alpha+1=0$
  - $\alpha^2=\alpha+1$

Example 5. GF(2^2) with polynomial α is the root of $x^2+x+1=0$

+	0	1	α	α^2
0	0	1	α	α^2
1	1	0	α^2	α
α	α	α^2	0	1
α^2	α^2	α	1	0

a	-a
0	0
1	1
α	α
α^2	α^2

X	0	1	α	α^2
0	0	0	0	0
1	0	1	α	α^2
α	0	α	α^2	α
α^2	0	α^2	1	1

a	a^{-1}
0	-
1	1
α	α^2
α^2	α

- Previous page's GF is made by polynomial $x^2+x+1=0$
- This polynomial is generation polynomial for GF.

Bit representation	Polynomial representation	Root index representation
00	0	$\alpha^{-\infty}$
01	1	$\alpha^0=1$
10	α	α^1
11	$\alpha+1$	α^2

- $\alpha^3= \alpha^2 X\alpha=(\alpha+1)\ \alpha= \alpha^2 + \alpha=1$

Example 6. GF(2^3) with polynomial

α is the root of $x^3+x+1=0$

Bit representation	Polynomial representation	Root index representation
000	0	$\alpha^{-\infty}$
001	1	$\alpha^0=1$
010	α	α^1
100	α^2	α^2
011	$\alpha+1$	α^3
110	$\alpha^2+\alpha$	α^4
111	$\alpha^2+\alpha+1$	α^5
101	α^2+1	α^6

- $\alpha^7= \alpha^3 X \alpha^3 X \alpha=(\alpha+1)(\alpha+1)\alpha= \alpha^3+\alpha=1$

+	0	1	α	α^2	$1+\alpha$	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	α^2+1
0	0	1	α	α^2	$1+\alpha$	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	α^2+1
1	1	0	$1+\alpha$	α^2+1	α	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	α^2
α	α	$\alpha+1$	0	$\alpha^2+\alpha$	1	α^2	α^2+1	$\alpha^2+\alpha+1$
α^2	α^2	α^2+1	$\alpha^2+\alpha$	0	$\alpha^2+\alpha+1$	α	$\alpha+1$	1
$1+\alpha$	$1+\alpha$	α	1	$\alpha^2+\alpha+1$	0	α^2+1	α^2	$\alpha^2+\alpha$
$\alpha^2+\alpha$	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	α^2	α	α^2+1	0	1	$1+\alpha$
$\alpha^2+\alpha+1$	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	α^2+1	$\alpha+1$	α^2	1	0	α
α^2+1	α^2+1	α^2	$\alpha^2+\alpha+1$	1	$\alpha^2+\alpha$	$1+\alpha$	α	0

Example 7. Simple Block Code

- 4bit information : 1011 (also shown as x^3+x+1)
- Make a simple block code as follows
  1. Shift 2 bit left 101100 ($x^5+x^3+x^2$)
  2. Calculate modulo by primitive polynomial x^2+x+1
     - **Ans=1**
  3. Add the modulo to 1. 101101 ($x^5+x^3+x^2+1$)
     - Now this code modulo(x^2+x+1)=0
- Send the 3. instead of 4bit information
- If received code's modulo(x^2+x+1) is 0,
  It is thought that No ERROR HAPPENED!

- In this example, the coefficient of the polynomial is 0 or 1, But, Reed Solomon code can handle more bits!

Example 7. Simple Block Code

1011

Information

$$A(x) = x^3 + x + 1$$

primitive polynomial

$$G(x) = x^2 + x + 1$$

$$\frac{A(x) \cdot x^2}{G(x)} = \frac{(x^3 + x + 1) \cdot x^2}{x^2 + x + 1} = \frac{x^5 + x^3 + x^2}{x^2 + x + 1} = \frac{(x^2 + x + 1)(x^3 + x^2 + x + 1) + 1}{x^2 + x + 1}$$

$$R(x) = 1$$

1011|01

Information parity

Code

$$W(x) = x^k A(x) + R(x) = G(x)Q(x) = x^5 + x^3 + x^2 + 1$$

Transmission

If the received code can be divided by G(x),
It is thought that No Error Happed.

Example 8. RS(5,3) code with GF(2^3)

- Remember GF(2^3) has 8 elements in Example 6.
- One element can handle 3bits.
- Reed Solomon (5,3) code has 3 information symbol + 2 parity symbol.
  - 3x3= 9bit information + 2x3=6bit parity

- Assume Information = $(1, \alpha、\alpha^2)$=(001 010 100)
  - $I(x)=x^2+\alpha x+\alpha^2$
  - $G(x)=x^2+\alpha^3x+\alpha$ -> $x^2=\alpha^3x+\alpha$
- $R(x)=(x^2I(x))\text{modulo}G(x)=(x^4+\alpha x^3+\alpha^2x^2)\ \text{modulo}G(x)$ $=\alpha^4x+1$
- $W(x)=x^2I(x)+R(x)= x^4+\alpha x^3+\alpha^2x^2 +\alpha^4x+1$
- RS(5,3) code = $(1, \alpha, \alpha^2 , \alpha^4 , 1)$=(001 010 100 110 001)

Calculation of R(x)

$$(x^4+\alpha x^3+\alpha^2x^2)\,\mathrm{mod}(x^2+\alpha^3x+\alpha)$$
$$=(\alpha^3x+\alpha)(\alpha^3x+\alpha)+\alpha x(\alpha^3x+\alpha)+\alpha^2(\alpha^3x+\alpha)$$
$$=\alpha^6x^2+\alpha^2+\alpha^4x^2+\alpha^2x+\alpha^5x+\alpha^3$$
$$=(\alpha^6+\alpha^4)x^2+(\alpha^2+\alpha^5)x+\alpha^2+\alpha^3$$
$$=\alpha^3(\alpha^3x+\alpha)+\alpha^3x+\alpha^5$$
$$=\alpha^6x+\alpha^4+\alpha^3x+\alpha^5$$
$$=\alpha^4x+1$$

RS code parameters

- Code length n ; $n \leqq q-1$, q=number of elements
- Information symbol length k : $k \leqq n-2t$
- Parity symbol length $c \leqq q-1-n+2t$
- Correctable symbol length = t

- Then, $q=2^3=8$
- Max n=7, when t=3, k=1, c=6
- R(5,3) case
  - n=5, q=8, k=3, Then max t = 1 only one symbol error correctable.

RS code error correction

- Error correction Decoding is more tough!
- Decoding is not covered in this lecture.