

Introduction to MIMO OFDM

Fire Tom Wada

Professor, Information Engineering, Univ. of the Ryukyus

Chief Scientist at Magna Design Net, Inc

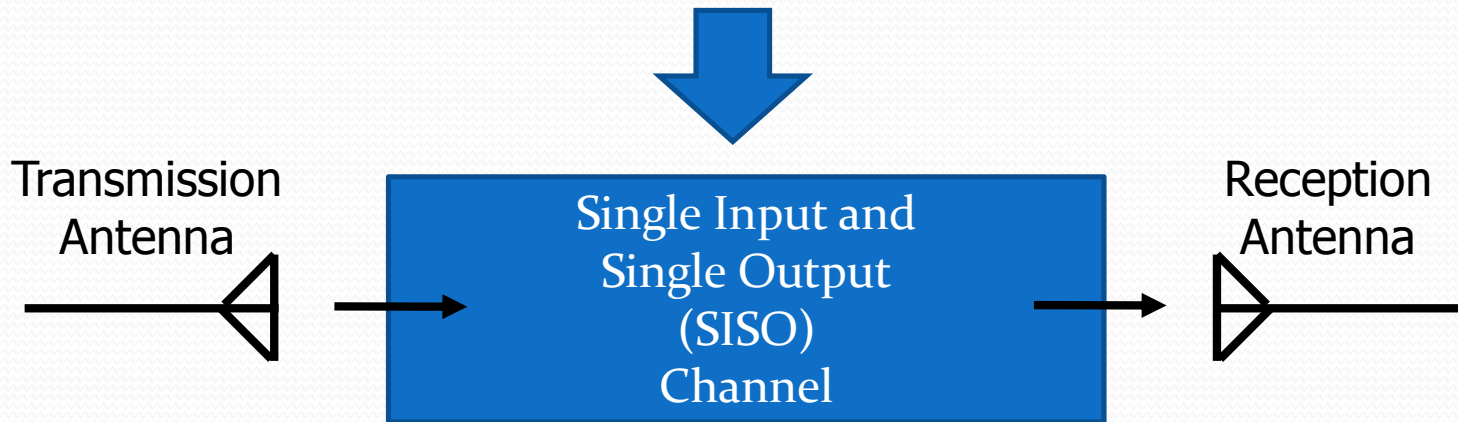
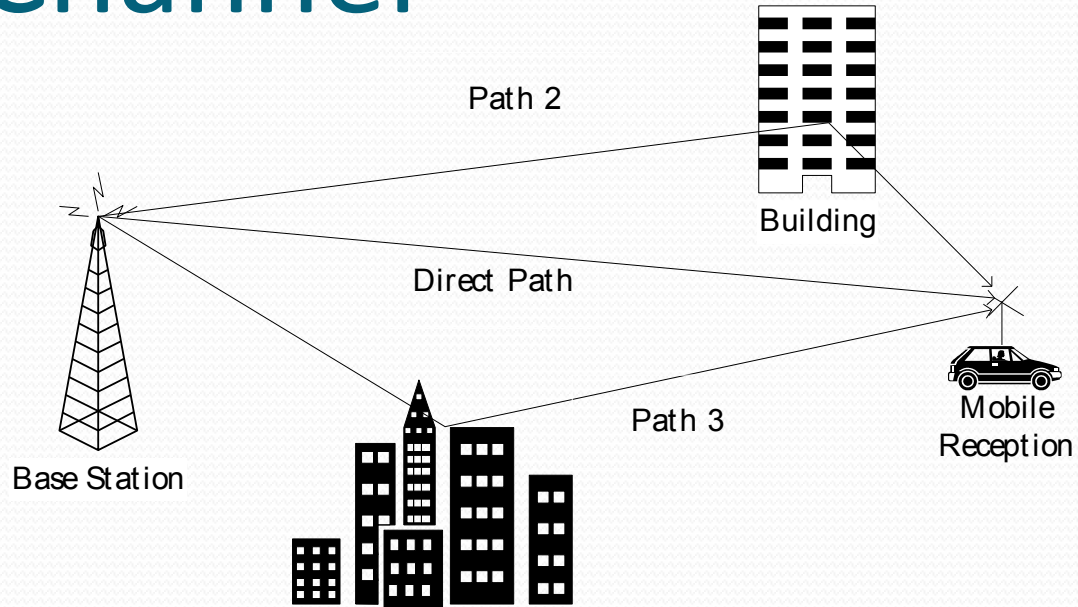
wada@ie.u-ryukyu.ac.jp

<http://www.ie.u-ryukyu.ac.jp/~wada/>

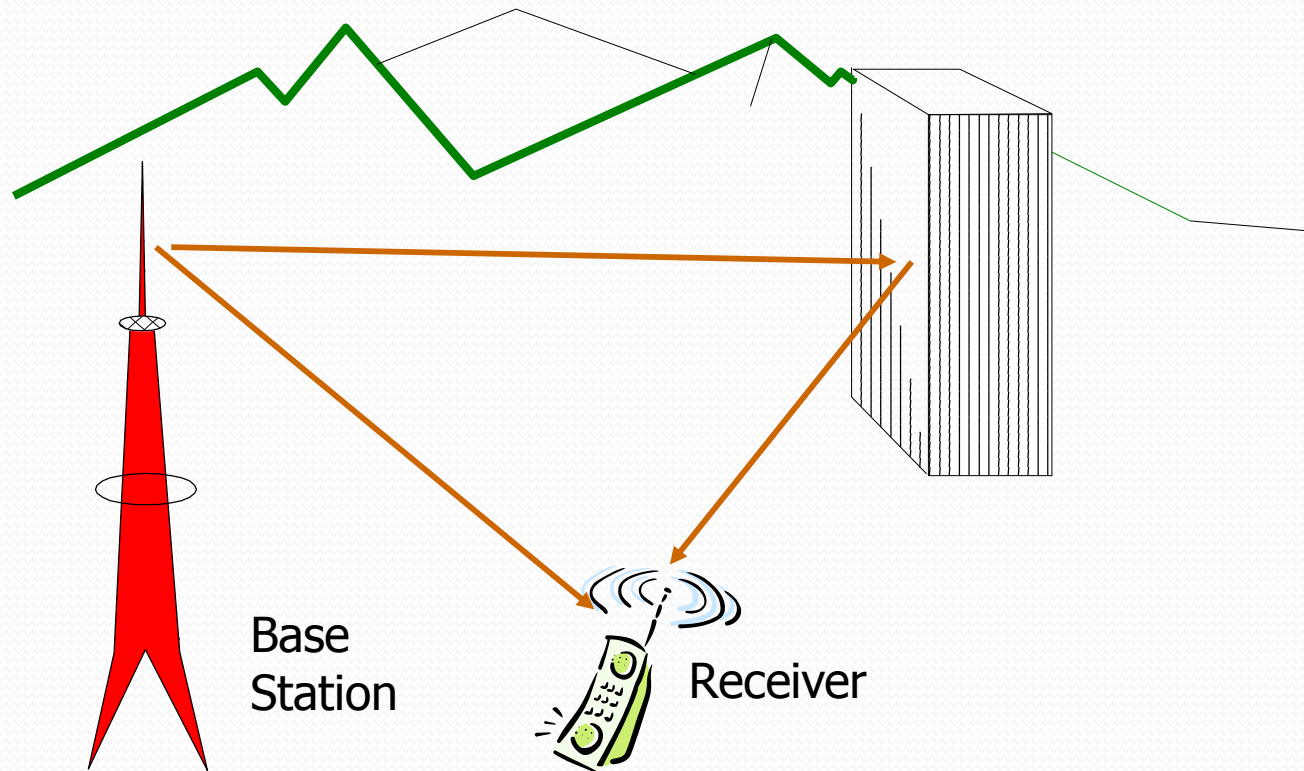
Section 1

- REVIEW

SISO Channel



Two path Multi path Channel Example



Channel Impulse Response = $[1, 0.5, 0, 0]$

Two path Multi path Channel Example

$$\begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix} = FFT * Channel * IFFT * \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}$$

$$\begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0.5 \\ 0.5 & 1 & 0 & 0 \\ 0 & 0.5 & 1 & 0 \\ 0 & 0 & 0.5 & 1 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}$$

$$\begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix} = \begin{pmatrix} H(0) & 0 & 0 & 0 \\ 0 & H(1) & 0 & 0 \\ 0 & 0 & H(2) & 0 \\ 0 & 0 & 0 & H(3) \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}$$

If time domain channel matrix is cyclic, Frequency Domain Channel Matrix is diagonal!

Additive Noise

$$\begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix} = FFT * \left[Channel * IFFT * \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix} + \begin{pmatrix} noise(0) \\ noise(1) \\ noise(2) \\ noise(3) \end{pmatrix} \right]$$

$$\begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0.5 \\ 0.5 & 1 & 0 & 0 \\ 0 & 0.5 & 1 & 0 \\ 0 & 0 & 0.5 & 1 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} \end{pmatrix} \begin{pmatrix} noise(0) \\ noise(1) \\ noise(2) \\ noise(3) \end{pmatrix}$$

$$\begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix} = \begin{pmatrix} H(0) & 0 & 0 & 0 \\ 0 & H(1) & 0 & 0 \\ 0 & 0 & H(2) & 0 \\ 0 & 0 & 0 & H(3) \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix} + \begin{pmatrix} N(0) \\ N(1) \\ N(2) \\ N(3) \end{pmatrix}$$

How to recover sending signal from receiver signal.

- EQUALIZE -

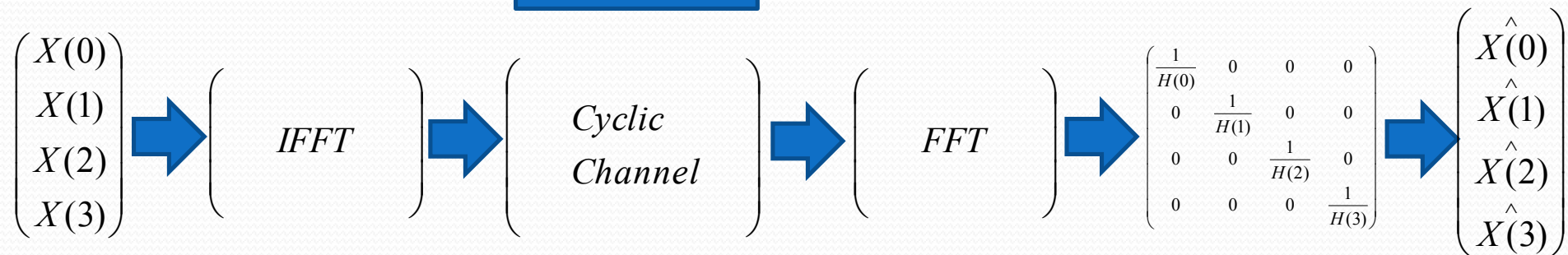
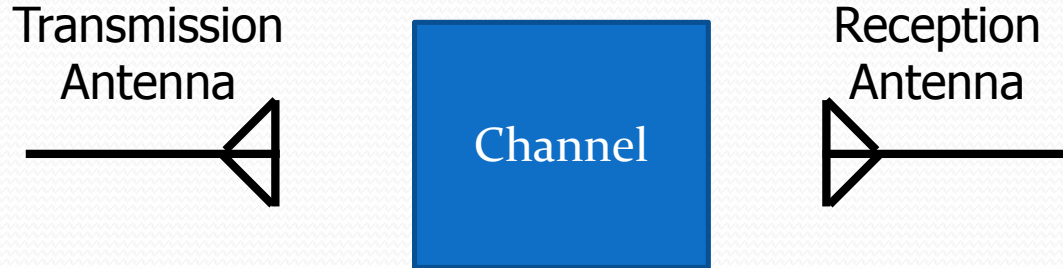
Ignore Noise

$$\begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix} = FFT * Channel * IFFT * \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix} = \begin{pmatrix} H(0) & 0 & 0 & 0 \\ 0 & H(1) & 0 & 0 \\ 0 & 0 & H(2) & 0 \\ 0 & 0 & 0 & H(3) \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}$$

Then

$$\begin{pmatrix} \hat{X}(0) \\ \hat{X}(1) \\ \hat{X}(2) \\ \hat{X}(3) \end{pmatrix} = \begin{pmatrix} \frac{1}{H(0)} & 0 & 0 & 0 \\ 0 & \frac{1}{H(1)} & 0 & 0 \\ 0 & 0 & \frac{1}{H(2)} & 0 \\ 0 & 0 & 0 & \frac{1}{H(3)} \end{pmatrix} \begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix}$$

Summary of Matrix model of OFDM



$$\begin{pmatrix} \hat{X}(0) \\ \hat{X}(1) \\ \hat{X}(2) \\ \hat{X}(3) \end{pmatrix} = \begin{pmatrix} \frac{1}{H(0)} & 0 & 0 & 0 \\ 0 & \frac{1}{H(1)} & 0 & 0 \\ 0 & 0 & \frac{1}{H(2)} & 0 \\ 0 & 0 & 0 & \frac{1}{H(3)} \end{pmatrix} \begin{matrix} \text{FFT} \\ \text{Cyclic} \\ \text{Channel} \\ \text{IFFT} \end{matrix} \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}$$

Important Mathematics

- Cyclic Matrix can be diagonalized by FFT and IFFT.
 - X^H is Hermitian of X , that is, complex conjugate and transpose.

$$\left(\begin{array}{c} FFT \\ \end{array} \right) \left(\begin{array}{c} \text{Cyclic} \\ \text{Matrix} \\ \end{array} \right) \left(\begin{array}{c} IFFT \\ \end{array} \right) = \begin{pmatrix} H(0) & 0 & 0 & 0 \\ 0 & H(1) & 0 & 0 \\ 0 & 0 & H(2) & 0 \\ 0 & 0 & 0 & H(3) \end{pmatrix} = \text{diag}(H(0), H(1), \dots)$$

Since

$$\left(\begin{array}{c} FFT \\ \end{array} \right)^H = \left(\begin{array}{c} IFFT \\ \end{array} \right) = U$$

$$U \left(\begin{array}{c} \text{Cyclic} \\ \text{Matrix} \\ \end{array} \right) U^H = \begin{pmatrix} H(0) & 0 & 0 & 0 \\ 0 & H(1) & 0 & 0 \\ 0 & 0 & H(2) & 0 \\ 0 & 0 & 0 & H(3) \end{pmatrix} = \text{diag}(H(0), H(1), \dots)$$

Unitary Matrix

- Unitary Matrix U can satisfy following property.

$$\mathbf{U}^H \mathbf{U} = \mathbf{U} \mathbf{U}^H = \mathbf{I}$$

when \mathbf{e}_i is vertical vector,

$$\mathbf{U} = [\mathbf{e}_1 \quad \cdots \quad \mathbf{e}_N]$$

then

$$\mathbf{e}_i^H \mathbf{e}_j = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

- Eigen value of Channel Cyclic Matrix is Channel Transfer Function as $(H(0), H(1), H(2), \dots)$.

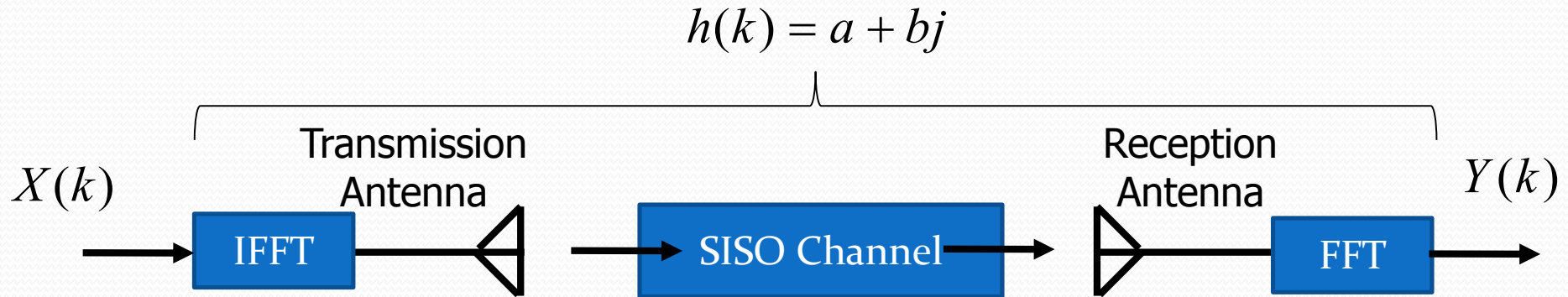
$$\left(\begin{array}{c} \text{Cyclic} \\ \text{Matrix} \end{array} \right) = U^H \begin{pmatrix} H(0) & 0 & 0 & 0 \\ 0 & H(1) & 0 & 0 \\ 0 & 0 & H(2) & 0 \\ 0 & 0 & 0 & H(3) \end{pmatrix} U$$

Section 2

- MIMO OFDM Channel Modeling

SISO Channel

- OFDM makes Multi-path channel simple complex $h(k)$ for freq= k .

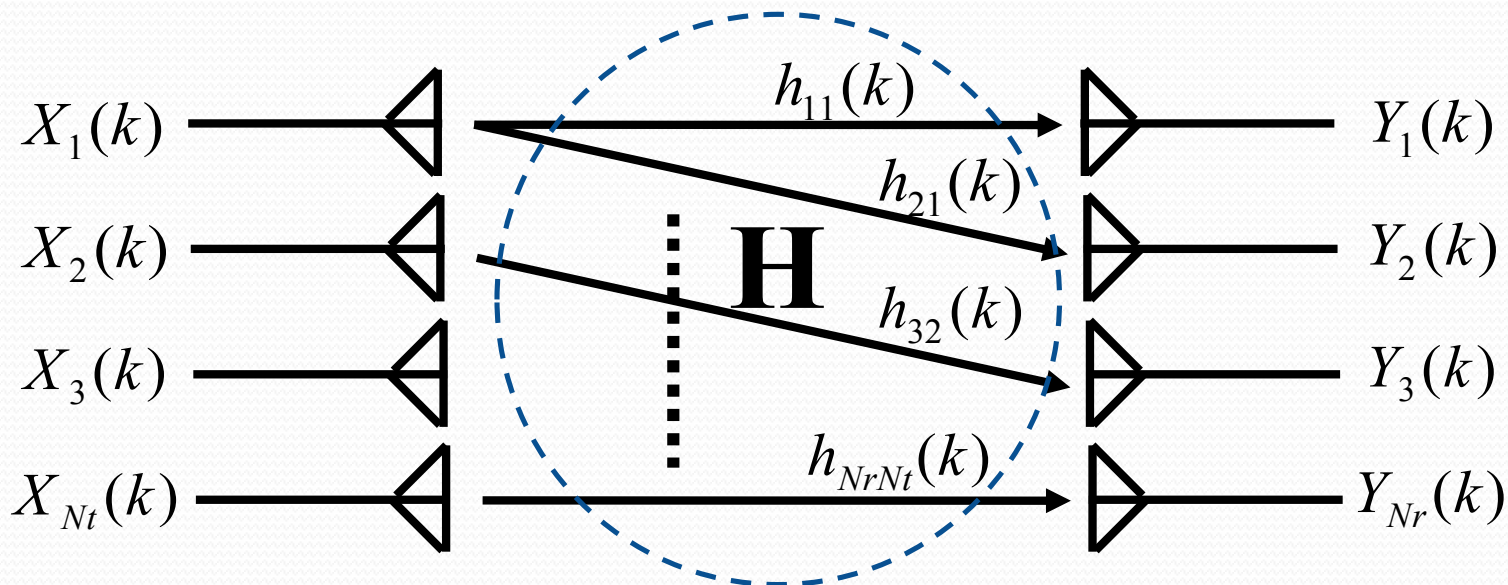


$$\begin{pmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{pmatrix} = FFT * Channel * IFFT * \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix} = \begin{pmatrix} h(0) & 0 & 0 & 0 \\ 0 & h(1) & 0 & 0 \\ 0 & 0 & h(2) & 0 \\ 0 & 0 & 0 & h(3) \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}$$

$$Y(k) = h(k)X(k)$$

MIMO Channel

- $N_r \times N_t$ SISO Channels for Freq= k -



$$\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{N_r} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & 0 & h_{1N_t} \\ h_{21} & h_{22} & 0 & h_{2N_t} \\ 0 & 0 & & \\ h_{N_r1} & h_{N_r2} & 0 & h_{N_rN_t} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N_t} \end{pmatrix} = \mathbf{H} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N_t} \end{pmatrix}$$

Singular value decomposition of $N_r \times N_t$ Matrix \mathbf{H}

- $N_r \times N_t$ matrix \mathbf{H} can be decomposed as below using $N_r \times N_r$ Unitary matrix \mathbf{V} and $N_t \times N_t$ Unitary matrix \mathbf{U} .
- Σ is $N_r \times N_t$ diagonal matrix.

$$\mathbf{H} = \mathbf{V}\Sigma\mathbf{U}^H$$

$$= \left[\begin{array}{c} \left[\begin{array}{ccccc} \sqrt{\lambda_1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_K} & 0 & 0 \end{array} \right] \\ Nr \times Nr \\ \left. \begin{array}{c} \\ \\ \end{array} \right] \left[\begin{array}{c} \\ \\ \\ \\ \end{array} \right] Nt \times Nt \right]$$

$\lambda_1, \dots, \lambda_K$ is eigen value of $\mathbf{H}^H \mathbf{H}$ and $\mathbf{H} \mathbf{H}^H$.

SVD Example by Matlab(1)

```
• H =  
•  
•     1     2     3  
•     2     4     5  
•  
• >> [U,S,V] = svd(H)  
• U =  
•    -0.4863    -0.8738  
•    -0.8738     0.4863  
• S =  
•  
•     7.6756         0         0  
•         0     0.2913         0  
• V =  
•    -0.2910     0.3396    -0.8944  
•    -0.5821     0.6791     0.4472  
•    -0.7593    -0.6508    -0.0000
```

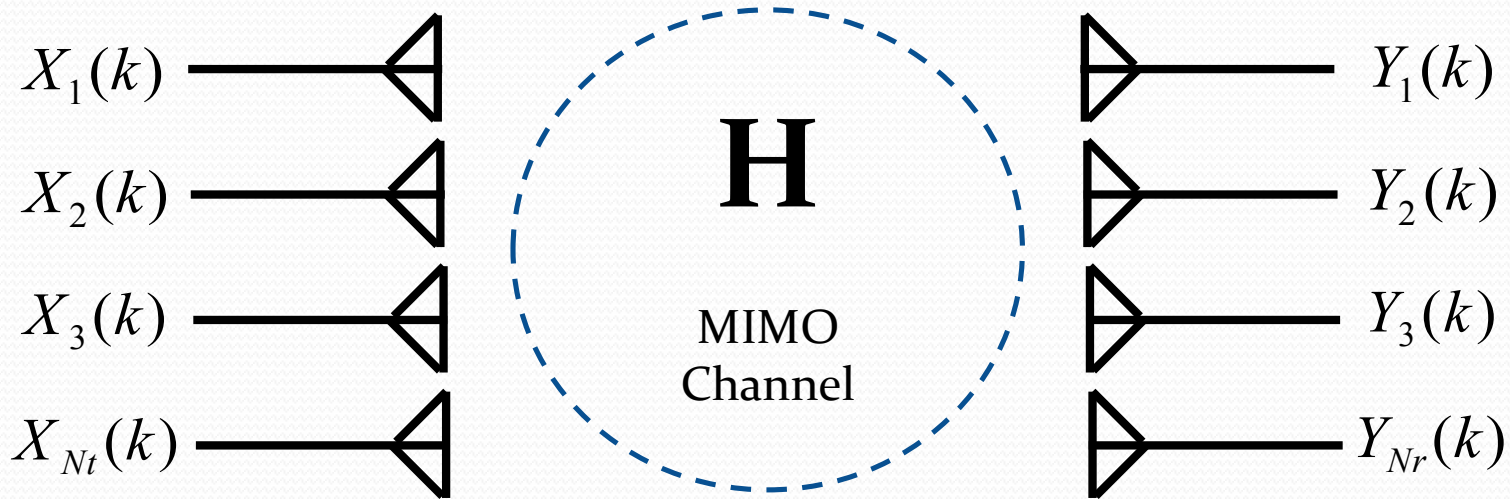
```
• H =  
•     1     2  
•     2     4  
•     3     5  
•  
• >> [U,S,V] = svd(H)  
• U =  
•  
•    -0.2910    -0.3396    -0.8944  
•    -0.5821    -0.6791     0.4472  
•    -0.7593     0.6508    -0.0000  
• S =  
•  
•     7.6756         0  
•         0     0.2913  
•         0         0  
• V =  
•  
•    -0.4863     0.8738  
•    -0.8738    -0.4863
```

SVD Example by Matlab(2)

```
• H =  
• 1.0000 + 1.0000i 2.0000 + 1.0000i  
• 1.0000 - 3.0000i 3.0000 - 1.0000i  
•  
• >> [U,S,V] = svd(H)  
• U =  
• -0.4616 - 0.0659i -0.4907 + 0.7361i  
• -0.3956 + 0.7913i -0.2863 - 0.3680i  
• S =  
• 5.0000 0  
• 0 1.4142  
• V =  
• -0.6594 0.7518  
• -0.5934 + 0.4616i -0.5205 + 0.4048i  
•  
• >> U*S*V'  
•  
• ans =  
•  
• 1.0000 + 1.0000i 2.0000 + 1.0000i  
• 1.0000 - 3.0000i 3.0000 - 1.0000i
```

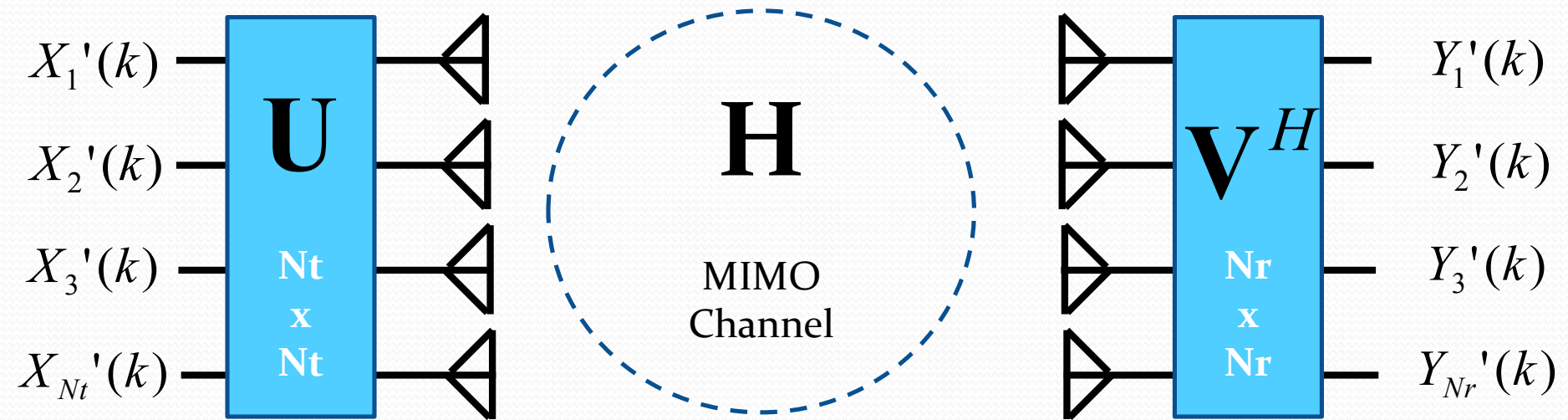
```
• >> U*S*V'  
•  
• ans =  
•  
• 1.0000 + 1.0000i 2.0000 + 1.0000i  
• 1.0000 - 3.0000i 3.0000 - 1.0000i  
•  
• >> U'*U  
• ans =  
•  
• 1.0000 0.0000 - 0.0000i  
• 0.0000 + 0.0000i 1.0000  
•  
• >> U*U'  
• ans =  
•  
• 1.0000 0.0000 - 0.0000i  
• 0.0000 + 0.0000i 1.0000
```


MIMO communication



$$\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{N_r} \end{pmatrix} = \mathbf{H} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N_t} \end{pmatrix}$$

Introduce pre-processing and post-processing



$$\begin{pmatrix} Y_1' \\ Y_2' \\ \vdots \\ Y_{Nr}' \end{pmatrix} = \mathbf{V}^H \mathbf{H} \mathbf{U} \begin{pmatrix} X_1' \\ X_2' \\ \vdots \\ X_{Nt}' \end{pmatrix}$$

There are $K(=\text{rank}(\mathbf{H}))$ independent channel

$$\begin{pmatrix} Y_1' \\ Y_2' \\ \vdots \\ Y_{Nr}' \end{pmatrix} = \mathbf{V}^H \mathbf{H} \mathbf{U} \begin{pmatrix} X_1' \\ X_2' \\ \vdots \\ X_{Nt}' \end{pmatrix}$$

$$\begin{pmatrix} Y_1' \\ Y_2' \\ \vdots \\ Y_{Nr}' \end{pmatrix} = \mathbf{V}^H \mathbf{V} \mathbf{\Sigma} \mathbf{U}^H \mathbf{U} \begin{pmatrix} X_1' \\ X_2' \\ \vdots \\ X_{Nt}' \end{pmatrix}$$

$$\begin{pmatrix} Y_1' \\ Y_2' \\ \vdots \\ Y_{Nr}' \end{pmatrix} = \mathbf{\Sigma} \begin{pmatrix} X_1' \\ X_2' \\ \vdots \\ X_{Nt}' \end{pmatrix}$$

$$\begin{pmatrix} Y_1' \\ Y_2' \\ \vdots \\ Y_{Nr}' \end{pmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sqrt{\lambda_K} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} X_1' \\ X_2' \\ \vdots \\ X_{Nt}' \end{pmatrix}$$

$$Y_1' = \sqrt{\lambda_1} \times X_1'$$

$$\vdots$$

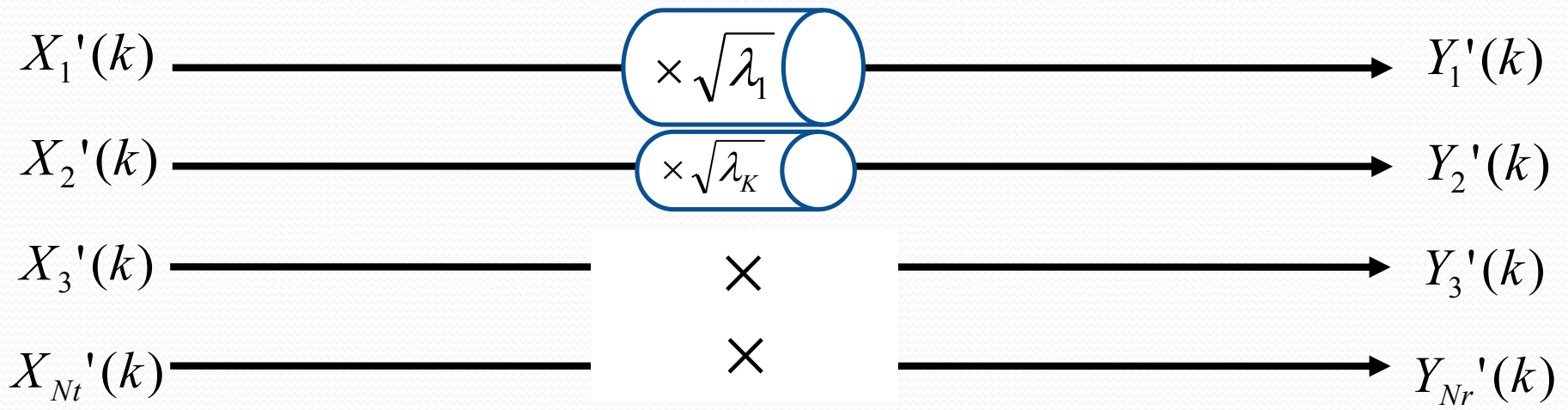
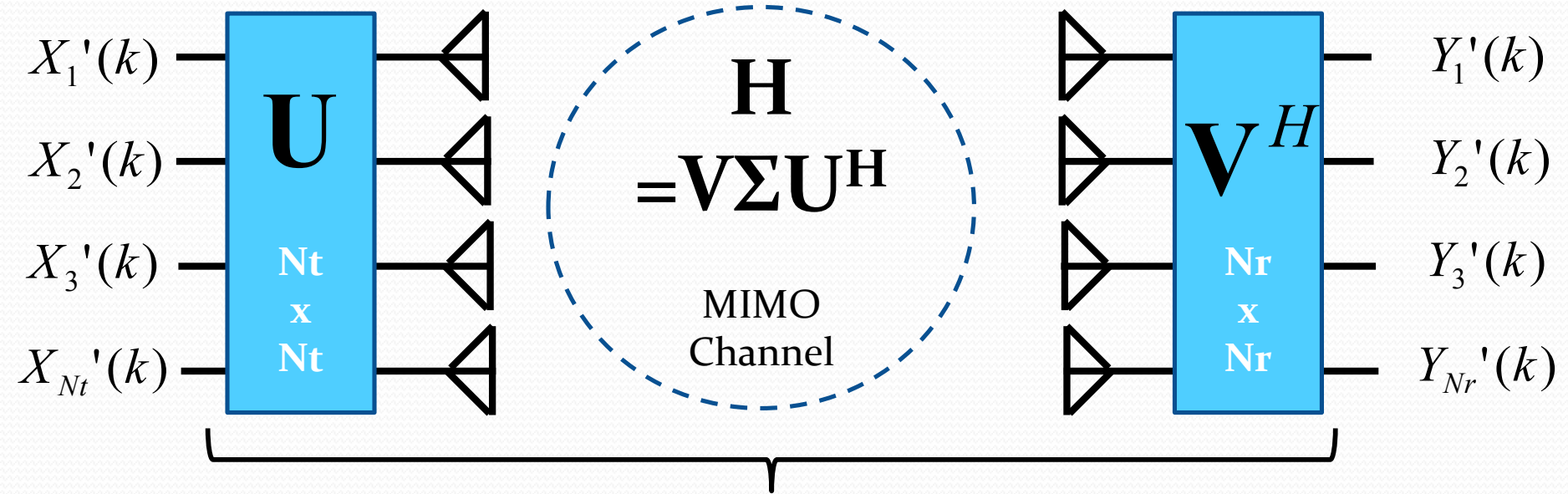
$$Y_K' = \sqrt{\lambda_K} \times X_K'$$

$$Y_{K+1}' = 0$$

$$\vdots$$

$$Y_{Nr}' = 0$$

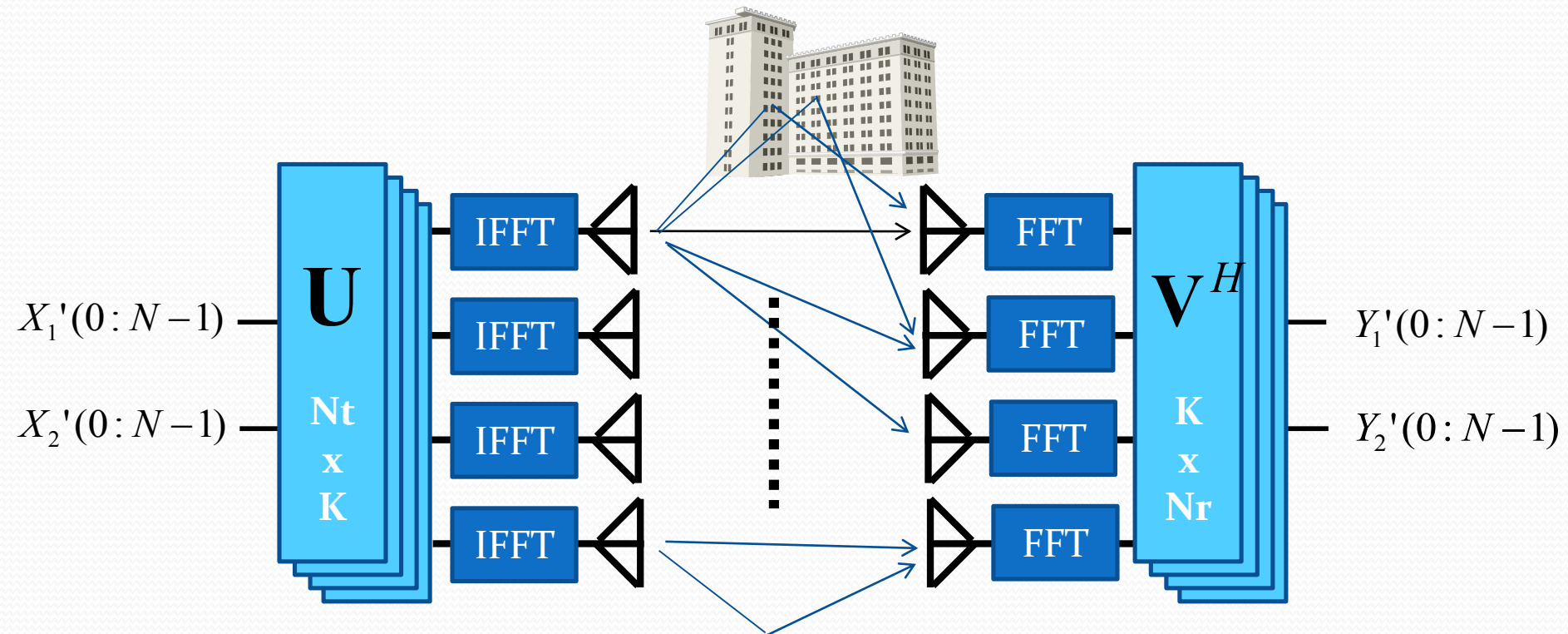
SVD-MIMO system



Put them altogether

MIMO-OFDM system

- Space Division Multiplexing by MIMO (K stream)
- Orthogonal Frequency Division Multiplexing (OFDM)



Summary

- This presentation shows matrix based modeling for both OFDM and MIMO and there are many similarity in mathematics.
 1. OFDM realizes many parallel communication channels in frequency domain.
 2. OFDM converts multi-path channel to simple one tap channel such as $h(k)=a+bj$ for Frequency= k .
 3. Then OFDM-based MIMO system can focus on simple channel matrix.
 4. By singular value decomposition (SVD), MIMO channel matrix H can be decomposed to $V*\Sigma*U^H$.
 5. Non-zero elements of Σ (rank of H) indicates parallel communication channel in space.